Logic Project

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Abstract

This document is a 3rd year undergraduate project in Logic, made by Aloïs Rosset at the University of Bristol and supervised by Dr Kentaro Fujimoto.

The aim is to pick up on the interest that Dr Fujimoto has raised in the last chapter of his lecture notes, which consist of an introduction to Gödel work and his two incompleteness theorems. It was unfortunate but the unit had to give only an overview of several notions, skipping some of them, because of the amount of time that is required to properly study this subject. Indeed the general statements and the proofs of this two famous results of the twentieth century consist of enough material to be taught as an entire academic unit.

One will go through the concepts of complete theory, representability in a theory, numeral-wise determination, effectivity, decidability, recursivity, recursive functions, sequence numbers, Gödel numbering, omega-consistency and some others.

Finally a few statement and their respective proofs of Gödel First Incompleteness Theorem will be given, in an increasing order of the quality of the assertion and of the difficulty of the demonstration.

The notations are the continuity of the aforementioned lecture notes and the new ones will be coherent within the existing context.

Contents

1 Introduction & Key Definitions

We are working with the full language of arithmetic $\mathcal{L}_{\mathbb{N}}$ and with the standard model of arithmetic $\mathfrak{N} = \{\mathbb{N}; 0, S, <, +, \cdot, E\}$. We are interested in the theory CnA_E where A_E consist of the following eleven sentences :

S1) $\forall x\ Sx \neq 0$
S2) $\forall x \forall y\ (Sx = Sy \rightarrow x = y)$
L1) $\forall x \forall y\ (x < Sy \leftrightarrow x \leqslant y)$
L2) $\forall x\ x \nless 0$
L3) $\forall x \forall y\ (x < y \vee x = y \vee y < x)$
A1) $\forall x\ x + 0 = x$
A2) $\forall x \forall y\ x + Sy = S(x + y)$
M1) $\forall x\ x \cdot 0 = 0$
M2) $\forall x \forall y\ x \cdot Sy = x \cdot y + x$
E1) $\forall x \forall y\ xE0 = \overline{1}$
E2) $\forall x \forall y\ xESy = xEy \cdot x$

Remember the notation $\overline{0} = 0$, $\overline{1} = S0$, and recursively $\overline{n} = S\overline{n-1} = S^n 0$
Note that $\mathfrak{N} \models A_E \Rightarrow$ Cn$A_E \subseteq$ ThA_E

Definition 1. A set $T \subseteq Sent_{\mathcal{L}}$ is a *theory* if it is closed under logical implication :

$$\forall \sigma \in Sent_{\mathcal{L}} : \quad T \models \sigma \;\Rightarrow\; \sigma \in T$$

Definition 2. A theory T is *complete* if for every $\sigma \in Sent_{\mathcal{L}_{\mathbb{N}}}$, either $\sigma \in T$ or $\neg\sigma \in T$

Notation 3. For $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$, with $FV(\phi) \subseteq \{v_1, \ldots, v_m\} \subseteq Var$ and $t_1, \ldots, t_k \in Tm_{\mathcal{L}_{\mathbb{N}}}$ (with $k \leqslant m$), we'll simplify the notation for substitution :

$$\phi(t_1, \ldots, t_k) = [\psi]\frac{t_1, \ldots, t_k}{v_1, \ldots, v_k} = [\ldots[[\psi]\frac{t_1}{v_1}]\ldots]\frac{t_k}{v_k}$$

where ψ is an appropriate alphabetic variant of ϕ s.t $t_1, \ldots, t_k$ are substitutable for $v_1, \ldots, v_k$ respectively in it.

Definition 4. A m-ary relation $R \subseteq \mathbb{N}^m$ is *defined* in $\mathfrak{N}$ by a formula $\rho \in Fml_{\mathcal{L}_{\mathbb{N}}}$, with $FV(\rho) \subseteq \{v_1, \ldots, v_m\} \subseteq Var$, iff $\forall a_1, \ldots, a_m \in \mathbb{N}$:

$$\begin{aligned} \langle a_1, \ldots, a_m \rangle \in R \; &:\Leftrightarrow \; \mathfrak{N} \models \rho[\![a_1, \ldots, a_m]\!] \\ &\Leftrightarrow \; \mathfrak{N} \models \rho(\overline{a_1}, \ldots, \overline{a_m}) \end{aligned}$$

The last equivalence was a revision exercise using the Substitution Lemma.

Definition 5. A m-ary relation $R \subseteq \mathbb{N}^m$ is *represented* in a theory T by a formula $\rho \in Fml_{\mathcal{L}_{\mathbb{N}}}$, with $FV(\rho) \subseteq \{v_1, \ldots, v_m\} \subseteq Var$, iff $\forall a_1, \ldots, a_m \in \mathbb{N}$:

$$\begin{aligned} \langle a_1, \ldots, a_m \rangle \in R \; &\Rightarrow \; \rho(\overline{a_1}, \ldots, \overline{a_m}) \in T \\ \langle a_1, \ldots, a_m \rangle \notin R \; &\Rightarrow \; \neg\rho(\overline{a_1}, \ldots, \overline{a_m}) \in T \end{aligned}$$

And when omitted later, the theory T will be CnA_E, meaning the following :

$$\begin{aligned} \langle a_1, \ldots, a_m \rangle \in R \; &\Rightarrow \; A_E \vdash \rho(\overline{a_1}, \ldots, \overline{a_m}) \\ \langle a_1, \ldots, a_m \rangle \notin R \; &\Rightarrow \; A_E \vdash \neg\rho(\overline{a_1}, \ldots, \overline{a_m}) \end{aligned}$$

Definition 6. A formula $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ with $FV(\rho) \subseteq \{v_1, \ldots, v_m\} \subseteq Var$ is *numeralwise determined* by A_E iff $\forall a_1, \ldots, a_m \in \mathbb{N}$

$$\begin{aligned} &\text{either } A_E \vdash \phi(\overline{a_1}, \ldots, \overline{a_m}) \\ &\quad \text{or } A_E \vdash \neg\phi(\overline{a_1}, \ldots, \overline{a_m}) \end{aligned}$$

Theorem 7. For $\rho \in Fml_{\mathcal{L}_{\mathbb{N}}}$ and a relation R :

$$\rho \text{ represents } R \Leftrightarrow \begin{array}{l} 1)\ \rho \text{ defines } R \text{ in } \mathfrak{N} \\ 2)\ \rho \text{ is numeralwise determined by } A_E \end{array}$$

Proof. $(\Rightarrow)$ ρ represent R and either $\langle a_1, \ldots, a_m \rangle \in R$ or $\langle a_1, \ldots, a_m \rangle \notin R$ holds, therefore the definition of 2) is fulfilled. And as $\mathfrak{N} \models A_E$:

$$\begin{array}{lllll} \langle a_1, \ldots, a_m \rangle \in R & \Rightarrow & A_E \vdash \rho(\overline{a_1}, \ldots, \overline{a_m}) & \Rightarrow & \mathfrak{N} \models \rho(\overline{a_1}, \ldots, \overline{a_m}) \\ \langle a_1, \ldots, a_m \rangle \notin R & \Rightarrow & A_E \vdash \neg\rho(\overline{a_1}, \ldots, \overline{a_m}) & \Rightarrow & \mathfrak{N} \not\models \rho(\overline{a_1}, \ldots, \overline{a_m}) \end{array}$$

$(\Leftarrow)$

$$\begin{aligned} \langle a_1, \ldots, a_m \rangle \in R &\overset{\text{by 1)}}{\Rightarrow} \mathfrak{N} \models \rho(\overline{a_1}, \ldots, \overline{a_m}) \\ &\overset{\mathfrak{N} \models A_E}{\Rightarrow} A_E \nvdash \neg\rho(\overline{a_1}, \ldots, \overline{a_m}) \\ &\overset{\text{by 2)}}{\Rightarrow} A_E \vdash \rho(\overline{a_1}, \ldots, \overline{a_m}) \end{aligned}$$

□

Lemma 8. For $k \in \mathbb{N}$:

$$A_E \vdash \forall x(x < \overline{k+1} \leftrightarrow x = \overline{0} \vee \cdots \vee x = \overline{k})$$

Proof. By induction on k :

- $k = 0$:

$$\begin{array}{lll} A_E \vdash \forall x \forall y(x < Sy \leftrightarrow x \leqslant y) & \text{by Ax(L1)} & \\ A_E \vdash x < S0 \leftrightarrow x \leqslant 0 & \text{by Right } \forall\text{-Elim twice} & \\ A_E \vdash x < \overline{1} \leftrightarrow x < 0 \vee x = 0 & \text{rewriting} & (1) \end{array}$$

So

$$\begin{array}{rll}
A_E, x = 0 \vdash x < 0 \vee x = 0 & \text{by Ax and Right } \vee\text{-Intro} & \\
A_E, x = 0 \vdash x < \overline{1} & \text{by (1) and MP} & \\
A_E \vdash x = 0 \to x < \overline{1} & \text{by Ded Thm} & (2)
\end{array}$$

$$\begin{array}{rll}
A_E, x < 0 \vdash x < 0 & \text{by Ax} & \\
A_E, x < 0 \vdash x \nless 0 & \text{by Ax(L2)} & \\
A_E, x < 0 \vdash x = 0 & \text{by Reductio} & \\
A_E, x = 0 \vdash x = 0 & \text{by Ax} & \\
A_E, x < 0 \vee x = 0 \vdash x = 0 & \text{by Left } \vee\text{-Intro} & \\
A_E \vdash x \leqslant 0 \to x = 0 & \text{by Ded Thm} & \\
A_E \vdash x < \overline{1} \to x = 0 & \text{by (1) and Transitivity of } \to & (3)
\end{array}$$

$$\begin{array}{rl}
A_E \vdash x < \overline{1} \leftrightarrow x = 0 & \text{by (2) and (3) and Right } \wedge\text{-Intro} \\
A_E \vdash \forall x(x < \overline{1} \leftrightarrow x = 0) & \text{by Gen}
\end{array}$$

- Suppose true for $k+1$.

$$\begin{array}{ll}
A_E \vdash x < \overline{k+2} \leftrightarrow x \leqslant \overline{k+1} & \text{by Ax(L2) \& Right } \forall\text{-Elim twice} \\
A_E \vdash x < \overline{k+2} \leftrightarrow x < \overline{k+1} \vee x = \overline{k+1} & \text{rewriting} \\
A_E \vdash x < \overline{k+2} \leftrightarrow x = \overline{0} \vee \ldots \vee x = \overline{k+1} \vee x = \overline{k+2} & \text{by I.H} \\
A_E \vdash \forall x(x < \overline{k+2} \leftrightarrow x = \overline{0} \vee \ldots \vee x = \overline{k+1} \vee x = \overline{k+2}) & \text{by Gen}
\end{array}$$

□

Lemma 9. For $t \in Tm_{\mathcal{L}_{\mathbb{N}}}$ closed, there exists a unique $n \in \mathbb{N}$ such that

$$A_E \vdash t = \overline{n}$$

Proof. Existence : By induction on $cp(t)$:

- $cp(t) = 0$: t is atomic means $t \in Var$ or t is a constant symbol. But t is supposed closed and the only constant symbol is 0, so we have $t = 0$. Take $n = 0$, and

$$A_E \vdash 0 = \overline{0} \qquad \text{by Ax(H7)}$$

- $t = St'$:

$$\begin{array}{ll}
A_E \vdash t' = \overline{m} & \text{some } m \in \mathbb{N} \text{ by I.H} \\
A_E \vdash t' = \overline{m} \to ([St' = Sx]\frac{t'}{x} \to [St' = Sx]\frac{\overline{m}}{x}) & \text{by Ax(H8)} \\
A_E \vdash St' = St' \to St' = S\overline{m} & \text{by MP} \\
A_E \vdash St' = St' & \text{by Ax(H7)} \\
A_E \vdash St' = \overline{m+1} & \text{by MP}
\end{array}$$

- $t = u + v$:

$A_E \vdash u = \overline{a}$	some $a \in \mathbb{N}$ by I.H
$A_E \vdash v = \overline{b}$	some $b \in \mathbb{N}$ by I.H
$A_E \vdash v = \overline{b} \to (t = u + v \to t = u + \overline{b})$	by Ax(H8)
$A_E \vdash t = u + \overline{b}$	by Ax(H7) and MP twice
$A_E \vdash u = \overline{a} \to (t = u + \overline{b} \to t = \overline{a} + \overline{b})$	by Ax(H8)
$A_E \vdash t = \overline{a} + \overline{b}$	by MP twice

Let's prove by induction on b that $A_E \vdash \overline{a} + \overline{b} = \overline{a + b}$

- $b = 0$:

$$A_E \vdash \overline{a} + 0 = \overline{a} \qquad \text{by Ax(A1)}$$

- $b = c + 1$:

$A_E \vdash \overline{a} + \overline{c} = \overline{a + c}$	by I.H
$A_E \vdash \overline{a} + S\overline{c} = S(\overline{a} + \overline{c})$	by Ax(A2)
$A_E \vdash \overline{a} + \overline{c} = \overline{a + c} \to (\overline{a} + S\overline{c} = S(\overline{a} + \overline{c}) \to \overline{a} + S\overline{c} = S(\overline{a + c}))$	by Ax(H8)
$A_E \vdash \overline{a} + S\overline{c} = S(\overline{a + c})$	by MP twice
$A_E \vdash \overline{a} + \overline{b} = \overline{a + b}$	by rewriting

Therefore $A_E \vdash t = \overline{a + b}$

- $t = u \cdot v$:

$A_E \vdash u = \overline{a}$	some $a \in \mathbb{N}$ by I.H
$A_E \vdash v = \overline{b}$	some $b \in \mathbb{N}$ by I.H
$A_E \vdash v = \overline{b} \to (t = u \cdot v \to t = u \cdot \overline{b})$	by Ax(H8)
$A_E \vdash t = u \cdot \overline{b}$	by Ax(H7) and MP twice
$A_E \vdash u = \overline{a} \to (t = u \cdot \overline{b} \to t = \overline{a} \cdot \overline{b})$	by Ax(H8)
$A_E \vdash t = \overline{a} \cdot \overline{b}$	by MP twice

Let's prove by induction on b that $A_E \vdash \overline{a} \cdot \overline{b} = \overline{a \cdot b}$

- $b = 0$:

$$A_E \vdash \overline{a} \cdot 0 = 0 \qquad \text{by Ax(M1)}$$

- $b = c + 1$:

$A_E \vdash \overline{a} \cdot \overline{c} = \overline{a \cdot c}$	by I.H
$A_E \vdash \overline{a} \cdot S\overline{c} = \overline{a} \cdot \overline{c} + \overline{a}$	by Ax(A2)
$A_E \vdash \overline{a} \cdot \overline{c} = \overline{a \cdot c} \to (\overline{a} \cdot S\overline{c} = \overline{a} \cdot \overline{c} + \overline{a} \to \overline{a} \cdot S\overline{c} = \overline{a \cdot c} + \overline{a}))$	by Ax(H8)
$A_E \vdash \overline{a} \cdot S\overline{c} = \overline{a \cdot c} + \overline{a}$	by MP twice
$A_E \vdash \overline{a} \cdot S\overline{c} = \overline{a \cdot c + a})$	by previous induction
$A_E \vdash \overline{a} \cdot \overline{b} = \overline{a \cdot b}$	by rewriting

Therefore $A_E \vdash t = \overline{a \cdot b}$

- $t = uEv$:

$$\begin{array}{ll} A_E \vdash u = \overline{a} & \text{some } a \in \mathbb{N} \text{ by I.H} \\ A_E \vdash v = \overline{b} & \text{some } b \in \mathbb{N} \text{ by I.H} \\ A_E \vdash v = \overline{b} \to (t = uEv \to t = uE\overline{b}) & \text{by Ax(H8)} \\ A_E \vdash t = u \cdot \overline{b} & \text{by Ax(H7) and MP twice} \\ A_E \vdash u = \overline{a} \to (t = uE\overline{b} \to t = \overline{a}E\overline{b}) & \text{by Ax(H8)} \\ A_E \vdash t = \overline{a}E\overline{b} & \text{by MP twice} \end{array}$$

Let's prove by induction on b that $A_E \vdash \overline{a}E\overline{b} = \overline{aEb}$

- $b = 0$:

$$A_E \vdash \overline{a}E\overline{0} = \overline{1} \qquad \text{by Ax(E1)}$$

- $b = c + 1$:

$$\begin{array}{ll} A_E \vdash \overline{a}E\overline{c} = \overline{aEc} & \text{by I.H} \\ A_E \vdash \overline{a}ES\overline{c} = \overline{a}E\overline{c} \cdot \overline{a} & \text{by Ax(A2)} \\ A_E \vdash \overline{a}E\overline{c} = \overline{aEc} \to (\overline{a}ES\overline{c} = \overline{a}E\overline{c} \cdot \overline{a} \to \overline{a}ES\overline{c} = \overline{aEc} \cdot \overline{a})) & \text{by Ax(H8)} \\ A_E \vdash \overline{a}ES\overline{c} = \overline{aEc} \cdot \overline{a} & \text{by MP twice} \\ A_E \vdash \overline{a}ES\overline{c} = \overline{aEc \cdot a}) & \text{by previous induction} \\ A_E \vdash \overline{a}E\overline{b} = \overline{aEb} & \text{by rewriting} \end{array}$$

Therefore $A_E \vdash t = \overline{aEb}$

□

Theorem 10. If $\tau \in Sent_{\mathcal{L}_\mathbb{N}}$ is quantifier-free with $\mathfrak{N} \models \tau$, then $A_E \vdash \tau$

Proof. By induction on $n := cp(\tau)$. Note that τ being a quantifier-free sentence means that terms in τ are closed : $var(\tau) = FV(\tau) = \varnothing$
Take t_1, t_2 closed terms $\overset{\text{Lemma 9}}{\Rightarrow}$ there are unique $n, m \in \mathbb{N}$: $A_E \vdash t_1 = \overline{n}$, $A_E \vdash t_2 = \overline{m}$

- τ is $t_1 = t_2$:
  If $\mathfrak{N} \models \tau \Rightarrow \mathfrak{N} \models \overline{n} = \overline{m}$
  Suppose $n \neq m \overset{\text{unicity}}{\Rightarrow} A_E \nvdash \overline{n} = \overline{m} \Rightarrow \mathfrak{N} \models \overline{n} \neq \overline{m}$ ↯
  So $n = m$ and $\overline{n} = \overline{m}$ and $A_E \vdash t_1 = t_2$
  If $\mathfrak{N} \models \neg\tau \Rightarrow \mathfrak{N} \models \overline{n} \neq \overline{m} \Rightarrow n \neq m$, w.l.o.g $n < m$.
  Let $p := m - n > 0 \overset{\text{(S1)}}{\Rightarrow} A_E \vdash \overline{p} \neq 0 \overset{\text{(S2)}}{\Rightarrow} A_E \vdash \overline{m} \neq \overline{n}$ i.e. $A_E \vdash \neg\tau$

- τ is $t_1 < t_2$:
  Lemma 8 $\Rightarrow A_E \vdash \overline{n} < \overline{m}$ iff $n < m$ and $A_E \vdash \overline{n} \nless \overline{m}$ iff $m \leqslant n$
  So

$$\mathfrak{N} \models \tau \Leftrightarrow n < m \Leftrightarrow A_E \vdash \tau$$
$$\mathfrak{N} \models \neg\tau \Leftrightarrow m \leqslant n \Leftrightarrow A_E \vdash \neg\tau$$

- $\tau = \neg\sigma$:

$$\mathfrak{N} \models \tau \Leftrightarrow \mathfrak{N} \models \neg\sigma \overset{\text{I.H}}{\Leftrightarrow} A_E \vdash \neg\sigma \Leftrightarrow A_E \vdash \tau$$
$$\mathfrak{N} \models \neg\tau \Leftrightarrow \mathfrak{N} \models \sigma \overset{\text{I.H}}{\Leftrightarrow} A_E \vdash \sigma \Leftrightarrow A_E \vdash \neg\neg\sigma \Leftrightarrow A_E \vdash \neg\tau$$

- $\tau = \sigma \to \lambda$:

$$\begin{aligned} \mathfrak{N} \models \tau \ &\Leftrightarrow\ \mathfrak{N} \models \neg\sigma \text{ or } \mathfrak{N} \vdash \lambda \overset{\text{I.H}}{\Leftrightarrow} A_E \vdash \neg\sigma \text{ or } A_E \vdash \lambda \\ &\Leftrightarrow\ A_E \vdash \neg\sigma \vee \lambda \ \Leftrightarrow\ A_E \vdash \tau \\ \mathfrak{N} \models \neg\tau \ &\Leftrightarrow \mathfrak{N} \models \sigma \text{ and } \mathfrak{N} \models \neg\lambda \overset{\text{I.H}}{\Leftrightarrow} A_E \vdash \sigma \text{ and } A_E \vdash \neg\lambda \\ &\Leftrightarrow A_E \vdash \sigma \wedge \neg\lambda \ \Leftrightarrow A_E \vdash \neg\tau \end{aligned}$$

□

Corollary 11. If $\tau \in Sent_{\mathcal{L}_{\mathbb{N}}}$ is an existential sentence true in $\mathfrak{N} \Rightarrow A_E \vdash \tau$

Proof. We have that $\tau = \exists x\rho$, some $x \in Var, \rho \in Sent_{\mathcal{L}_{\mathbb{N}}}$

$$\begin{aligned} \mathfrak{N} \models \tau &\Rightarrow \text{there exists } n \in \mathbb{N} : \mathfrak{N}, s\tfrac{n}{x} \models \rho \text{ (s an arbitrary variable assignement)} \\ &\Rightarrow \mathfrak{N}, s\tfrac{\bar{s}(\bar{n})}{x} \models \rho \\ &\Rightarrow \mathfrak{N} \models \rho(\bar{n}) \text{ by the Substitution Lemma} \end{aligned}$$

And say w.l.o.g that $\rho(\bar{n})$ is quantifier-free, then by Theorem 10 :

$$\begin{aligned} &A_E \vdash \rho(\bar{n}) \\ &A_E \vdash \exists x\rho \qquad\qquad \text{by Right } \exists\text{-Intro} \end{aligned}$$

□

Theorem 12. a) Atomic formula are numeralwise determined by A_E
b) ϕ, ψ are numeralwise determined by $A_E \Rightarrow$ so are $\neg\phi$ and $\phi \to \psi$
c) ϕ is numeralwise determined by $A_E \Rightarrow$ so are

$$\forall x(x < y \to \phi)$$
$$\exists x(x < y \wedge \phi)$$

Proof. a) Let $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ be atomic. We have $FV(\phi) \subseteq \{v_1, \ldots, v_n\}$ for some $n \in \mathbb{N}$. Take $t_1, \ldots, t_n \in Tm_{\mathcal{L}_{\mathbb{N}}}$ closed. As $\phi(t_1, \ldots, t_n)$ is a quantifier-free sentence, we have

$$\begin{aligned} \mathfrak{N} \models \phi(t_1, \ldots, t_n) &\overset{\text{Theorem 10}}{\Rightarrow} A_E \vdash \phi(t_1, \ldots, t_n) \\ &\overset{\text{Right } \exists\text{-Intro}}{\Rightarrow} A_E \vdash \exists v_1 \ldots \exists v_n \phi \\ &\overset{\text{Corollary 11}}{\Rightarrow} A_E \vdash \phi \\ \mathfrak{N} \models \neg\phi(t_1, \ldots, t_n) &\overset{\text{Theorem 10}}{\Rightarrow} A_E \vdash \neg\phi(t_1, \ldots, t_n) \\ &\overset{\text{Right } \exists\text{-Intro}}{\Rightarrow} A_E \vdash \exists v_1 \ldots \exists v_n \neg\phi \\ &\overset{\text{Corollary 11}}{\Rightarrow} A_E \vdash \neg\phi \end{aligned}$$

b) Let $\phi, \psi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ be numeralwise determined by A_E. Notice that

$$\begin{aligned} \text{either } A_E \vdash \phi &\Rightarrow A_E \vdash \neg(\neg\phi) \\ \text{or } A_E \vdash \neg\phi &\Rightarrow A_E \vdash (\neg\phi) \end{aligned}$$

So $\neg\phi$ is numeralwise determined. And by looking at cases :

- If $A_E \vdash \psi$, then

$$\begin{aligned} A_E, \phi &\vdash \psi && \text{by Monicity} \\ A_E &\vdash \phi \to \psi && \text{by Ded Thm} \end{aligned}$$

- If $A_E \vdash \neg\psi$ and $A_E \vdash \phi$, then

$$\begin{aligned} A_E, \phi \to \psi &\vdash \phi \to \psi && \text{by Ax} \\ A_E, \phi \to \psi &\vdash \neg\psi \to \neg\phi && \text{by Contrapositive} \\ A_E, \phi \to \psi &\vdash \neg\psi && \text{by Hypothesis} \\ A_E, \phi \to \psi &\vdash \neg\phi && \text{by MP} \\ A_E, \phi \to \psi &\vdash \phi && \text{by Hypothesis} \\ A_E &\vdash \neg(\phi \to \psi) && \text{by Reductio} \end{aligned}$$

- If $A_E \vdash \neg\psi$ and $A_E \vdash \neg\phi$, then

$$\begin{aligned} A_E, \neg(\phi \to \psi) &\vdash \neg(\phi \to \psi) && \text{by Ax} \\ A_E, \neg(\phi \to \psi) &\vdash \neg\phi && \text{by Hypothesis} \\ A_E, \neg(\phi \to \psi), \neg\psi &\vdash \neg\phi && \text{by Monicity} \\ A_E, \neg(\phi \to \psi) &\vdash \neg\psi \to \neg\phi && \text{by Ded Thm} \\ A_E, \neg(\phi \to \psi) &\vdash \phi \to \psi && \text{by Contrapositive} \\ A_E &\vdash \phi \to \psi && \text{by Reductio} \end{aligned}$$

So $\phi \to \psi$ is numeralwise determined.

c) Let $\phi, \psi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ be numeralwise determined by A_E. We have $FV(\phi) \subseteq \{x, y, v_1, \ldots, v_n\}$ for some $n \in \mathbb{N}$. Take arbitrary $a, b_1, \ldots, b_n \in \mathbb{N}$. Let $\theta = \exists x(x < y \wedge \phi)$. To show :

$$\begin{aligned} \text{either } &A_E \vdash \theta \\ \text{or } &A_E \vdash \neg\theta \end{aligned}$$

But observe that $A_E \vdash \theta \Leftrightarrow A_E \vdash \theta(\overline{a}, \overline{b_1}, \ldots, \overline{b_n})$

($\Rightarrow$)

Suppose $A_E \vdash \theta$

$$\begin{aligned} A_E &\vdash \forall x \forall v_1 \ldots \forall v_n \theta && \text{by Gen} \\ A_E &\vdash \theta(\overline{a}, \overline{b_1}, \ldots, \overline{b_n}) && \text{by Right } \forall\text{-Intro} \end{aligned}$$

($\Leftarrow$)

Suppose $A_E \vdash \theta(\overline{a}, \overline{b_1}, \ldots, \overline{b_n})$

$$\begin{aligned} A_E &\vdash \exists x \exists v_1 \ldots \exists v_n \phi && \text{by Right } \exists\text{-Intro} \\ A_E &\vdash \theta && \text{by Corollary 11} \end{aligned}$$

Now, to prove what we aim to, we divide in two cases :

- If $\mathfrak{N} \models \exists x(x < \overline{a} \wedge \phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$ then there exists $c \in \mathbb{N}$ with $c < a$ and $\mathfrak{N}, s\frac{c}{x} \models \phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$ (s an arbitrary var. assign.).
$\overset{\text{Subst Lem}}{\Rightarrow}$ there exists $c \in \mathbb{N}$ with $c < a$ and $\mathfrak{N} \models \phi(\overline{c}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$.
As ϕ is numeralwise det. $\Rightarrow$ so is $\phi(\overline{c}, \overline{a}, \overline{b_1} \ldots, \overline{b_n})$ as shown above, so

$$A_E \vdash \phi(\overline{c}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$$

And $c < a \Rightarrow \mathfrak{N} \models \overline{c} < \overline{a}$, so

$$\begin{array}{ll} A_E \vdash \overline{c} < \overline{a} & \text{by (a) as it's an atomic formula} \\ A_E \vdash \overline{c} < \overline{a} \wedge \phi(\overline{c}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) & \text{by Right } \wedge\text{-Intro} \\ A_E \vdash [x < \overline{a} \wedge \phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})]\frac{\overline{c}}{x} & \text{rewriting} \\ A_E \vdash \exists x(x < \overline{a} \wedge \phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})) & \text{by Right } \exists\text{-Intro} \\ A_E \vdash \theta(\overline{a}, \overline{b_1}, \ldots, \overline{b_n}) & \end{array}$$

- If $\mathfrak{N} \models \neg\exists x(x < \overline{a} \wedge \phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$, i.e

$$\mathfrak{N} \models \forall x(x < \overline{a} \rightarrow \neg\phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$$

then as before, by the Substitution Lemma we have

$$\begin{array}{l} \text{for all } c < a : \mathfrak{N} \models \neg\phi(\overline{c}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) \\ \overset{(b)}{\Rightarrow} \text{for all } c < a : A_E \vdash \neg\phi(\overline{c}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) \end{array}$$

And remember that by Lemma 8 :

$$A_E \vdash \forall x(x < \overline{a} \rightarrow x = \overline{0} \vee \ldots \vee \overline{a-1})$$

Now if $\mathfrak{M}, s$ is a structure s.t : $\mathfrak{M}, s \models A_E \cup \{x = \overline{0} \vee \ldots \vee x = \overline{a-1}\}$, then

$$\begin{array}{lll} s(x) = 0, 1, \ldots, a-1 \Rightarrow s(x) < a & & \\ & \Rightarrow A_E \vdash \neg\phi(\overline{s(x)}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) & \text{as shown just before} \\ & \Rightarrow \mathfrak{M}, s \models \neg\phi(\overline{s(x)}, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) & \text{since } \mathfrak{M}, s \models A_E \\ & \Rightarrow \mathfrak{M}, s \models \neg\phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) & \text{by Subst Lem} \end{array}$$

All this shows

$$A_E, x = \overline{0} \vee \ldots \vee x = \overline{a-1} \vdash \neg\phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})$$

$$A_E \vdash x = \overline{0} \vee \ldots \vee x = \overline{a-1} \rightarrow \neg\phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n}) \quad \text{by Ded Thm}$$

Therefore

$$\begin{array}{l} A_E \vdash \forall x(x < \overline{a} \rightarrow \neg\phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})) \\ A_E \vdash \neg\exists x(x < \overline{a} \wedge \neg\phi(x, \overline{a}, \overline{b_1}, \ldots, \overline{b_n})) \\ A_E \vdash \neg\theta(\overline{a}, \overline{b_1}, \ldots, \overline{b_n}) \end{array}$$

□

Definition 13. A function $F : \mathbb{N}^m \rightarrow \mathbb{N}$ is *functionally representable* (in CnA_E) by ϕ, where $FV(\phi) \subseteq \{v_1, \ldots, v_{m+1}\} \subseteq Var$, iff $\forall a_1, \ldots, a_m \in \mathbb{N}$:

$$A_E \vdash \forall v_{m+1}(\phi(\overline{a_1}, \ldots, \overline{a_m}, v_{m+1}) \leftrightarrow v_{m+1} = \overline{f(a_1, \ldots, a_m)})$$

2 Church-Turing Approach

Definition 14. A procedure is *effective* iff there is exact instructions of finite length that even a computing machine could understand, execute and always give a precise answer (like 'yes or 'no') after a finite period of time (without restriction on the order of magnitude)

Definition 15. A set Σ of expressions is *decidable* iff there exists an effective procedure checking that a given element α belongs or not to Σ

Definition 16. A theory T is *(finitely) axiomatizable* iff there is a decidable (finite) set $\Sigma \subseteq Sent_{\mathcal{L}_{\mathbb{N}}}$ s.t. $T = Cn\Sigma$

Definition 17. A relation R on $\mathbb{N}$ is *recursive* iff it is representable in some consistent finitely axiomatizable theory

Note. A_E being finite is decidable, thus CnA_E is finitely axiomatizable and CnA_E being satisfiable in $\mathfrak{N}$ implies it is consistent. Therefore a relation representable in CnA_E is recursive

Definition 18. A set A of expressions is *effectively enumerable* iff there exists an effective procedure listing the members of A. If A is infinite, a specific element must still always appear on the list at some point.

We give the following theorem without a proof, which isn't hard but requires some logic not covered in the Logic unit.

Theorem 19. Any relation representable in a consistent axiomatizable theory is decidable.

Corollary 20. A recursive relation is decidable

Now Church-Turing Thesis state that the converse is also true.

Church-Turing Thesis. For a relation R : R is recursive iff R is decidable.

This isn't a mathematical theorem but everything tends to support its veracity. The main problem is the informal notion if decidability, which is sort of a natural way to approach complex concepts. Empirical fact : as of today, every relation perceived as decidable has been proved to be recursive.

3 Representability Tools

Theorem 21. A formula ϕ functionally represents a function f $\Rightarrow$ it represents R_f

Proof. Let's prove the statement in the case where f is a m-ary function symbol with $m = 1$. We have :

$$
\begin{aligned}
&A_E \vdash \forall b(\phi(\overline{a}, b) \leftrightarrow b = \overline{f(a)}) && \text{by Hypothesis} \\
&A_E \vdash \phi(\overline{a}, \overline{b}) \leftrightarrow \overline{b} = \overline{f(a)} && \text{by Right } \forall\text{-Elimination, since} \\
& && \quad \overline{b} \text{ is closed thus substitutable}
\end{aligned}
$$

Therefore :

$$
\begin{aligned}
\langle a, b\rangle \in R_f &\Rightarrow f(a) = b \\
&\Rightarrow \overline{f(a)} = \overline{b} \\
&\Rightarrow A_E \vdash \phi(\overline{a}, \overline{b}) \\
\langle a, b\rangle \notin R_f &\Rightarrow \neg(f(a) = b) \\
&\Rightarrow \neg(\overline{f(a)} = \overline{b}) \\
&\Rightarrow A_E \vdash \neg\phi(\overline{a}, \overline{b})
\end{aligned}
$$

□

Theorem 22. A graph of a function R_f is representable $\Rightarrow$ there exists ϕ functionally representing f

Proof. Let's prove the statement in the case where f is a m-ary function symbol with $m = 1$. Suppose $\theta \in Fml_{\mathcal{L}_{\mathbb{N}}}$ represents R_f and let

$$\phi := \theta(v_1, v_2) \wedge \forall z(z < v_2 \rightarrow \neg\theta(v_1, z))$$

($\rightarrow$) Let $\Gamma = A_E \cup \{\phi(\overline{a}, v_2)\}$

$$
\begin{aligned}
A_E &\vdash \theta(\overline{a}, \overline{f(a)}) && \text{since } \langle a, f(a)\rangle \in R_f && (4) \\
\Gamma &\vdash \theta(\overline{a}, \overline{f(a)}) && \text{by Monicity} \\
\Gamma &\vdash \phi(\overline{a}, v_2) && \text{by Ax} \\
\Gamma &\vdash \forall z(z < v_2 \rightarrow \neg\theta(\overline{a}, z)) && \text{by Right } \wedge\text{-Elim} \\
\Gamma &\vdash \overline{f(a)} < v_2 \rightarrow \neg\theta(\overline{a}, \overline{f(a)}) && \text{by Right } \forall\text{-Elim} \\
\Gamma &\vdash \theta(\overline{a}, \overline{f(a)}) \rightarrow \neg\overline{f(a)} < v_2 && \text{by Contrapositive} \\
\Gamma &\vdash \neg\overline{f(a)} < v_2 && \text{by MP}
\end{aligned}
$$

$A_E, v_2 = \overline{b} \vdash \neg\theta(\overline{a}, v_2)$	for all $b = 0, \ldots, f(a) - 1$, since $b < f(a) \Rightarrow \langle a, b \rangle \notin R_f$
$A_E, v_2 = \overline{0} \vee \ldots \vee v_2 = \overline{f(a) - 1}$ $\vdash \neg\theta(\overline{a}, v_2)$	by Left $\vee$-Intro
$A_E \vdash v_2 = \overline{0} \vee \ldots \vee v_2 = \overline{f(a) - 1} \to \neg\theta(\overline{a}, v_2)$	by Ded Thm
$A_E \vdash v_2 < \overline{f(a)} \to v_2 = \overline{0} \vee \ldots \vee v_2 = \overline{f(a) - 1}$	by Lemma 8 + Right $\forall$-Elim
$A_E \vdash v_2 < \overline{f(a)} \to \neg\theta(\overline{a}, v_2)$	by Transitivity of $\to$ (5)
$\Gamma \vdash v_2 < \overline{f(a)} \to \neg\theta(\overline{a}, v_2)$	by Monicity
$\Gamma \vdash \theta(\overline{a}, v_2) \to \neg v_2 < \overline{f(a)}$	by Contrapositive
$\Gamma \vdash \theta(\overline{a}, v_2)$	by Ax and Right $\wedge$-Elim
$\Gamma \vdash \neg v_2 < \overline{f(a)}$	by MP

$\Gamma \vdash v_2 < \overline{f(a)} \vee v_2 = \overline{f(a)} \vee \overline{f(a)} < v_2$	by Ax, it's (L3)
$\Gamma \vdash v_2 = \overline{f(a)}$	by elimination
$A_E \vdash \phi(\overline{a}, v_2) \to v_2 = \overline{f(a)}$	by Ded Thm
$A_E \vdash \forall v_2(\phi(\overline{a}, v_2) \to v_2 = \overline{f(a)})$	by Gen

$(\leftarrow)$

$A_E \vdash \theta(\overline{a}, \overline{f(a)})$	by (4)
$A_E \vdash v_2 < \overline{f(a)} \to \neg\theta(\overline{a}, v_2)$	by (5)
$A_E \vdash \phi(\overline{a}, \overline{f(a)})$	by Right $\wedge$-Intro
$A_E \vdash \forall v_2(v_2 = \overline{f(a)} \to \phi(\overline{a}, v_2))$	since they're logically equiv.

□

Definition 23. Recursive functions is a sub-class of $\{F : \mathbb{N}^k \to \mathbb{N} : k \in \mathbb{N}\}$ obtained recursively by a basic set of functions containing :

- successor function $S(n) := n + 1$
- addition $+(n, m) := n + m$
- multiplication $\cdot(n, m) := n \cdot m$
- exponentiation $E(n, m) := n^m$
- projective functions $I_i^m(n_1, \ldots, n_m) := n_i$
- characteristic function of the $<$ relation $\chi_<(n, m) := \begin{cases} 1 \text{ if } n < m \\ 0 \text{ otherwise} \end{cases}$
- characteristic function of the $=$ relation $\chi_=(n, m) := \begin{cases} 1 \text{ if } n = m \\ 0 \text{ otherwise} \end{cases}$

that is closed under :

- **Substitution :** $g, h_1, \ldots, h_n$ are recursive functions
  $\Rightarrow f(a_1, \ldots, a_m) := g(h_1(a_1, \ldots, a_m), \ldots, h_n(a_1, \ldots, a_m))$ is a recursive function

- **Minimalization :** g (n+1)-ary recursive function and $\forall a_1, \ldots, a_n \exists b : g(a_1, \ldots, a_n, b) = 0$
  $\Rightarrow \mu b[g(\vec{a}, b) = 0] := min\{b \mid g(\vec{a}, b) = 0\}$ is a recursive function

Theorem 24. 1. Basic recursive functions are functionally representable

2. $g, h_1, \ldots, h_n$ are functionally representable
   $\Rightarrow f(a_1, \ldots, a_m) := g(h_1(a_1, \ldots, a_m), \ldots, h_n(a_1, \ldots, a_m))$ is functionally representable

3. g (n+1)-ary function, functionally representable and $\forall a_1, \ldots, a_n \exists b : g(a_1, \ldots, a_n, b) = 0$
   $\Rightarrow \mu b[g(\vec{a}, b) = 0]$ is functionally representable

And therefore all recursive functions are functionally representative in CnA_E and have a recursive graph.

Proof. 1. Observe that when a graph R_f of a function f is defined by an equation of the form $v_{m+1} = t$ where $t \in Tm_{\mathcal{L}_\mathbb{N}}$ and $var(t) \subseteq \{v_1, \ldots, v_m\}$, then R_f is representable by Theorem 7. This follows from the fact that an equation is numeralwise defined as an atomic formula by Theorem 12. And in fact the equation even functionally represents R_f since

$$\forall v_{m+1}(v_{m+1} = t(\overline{a_1}, \ldots, \overline{a_m}) \leftrightarrow v_{m+1} = \overline{f(a_1, \ldots, a_m)})$$

is, clearly by the definition of a structure, logically equivalent to

$$t(\overline{a_1}, \ldots, \overline{a_m}) = \overline{f(a_1, \ldots, a_m)}$$

and the latter is true in $\mathfrak{N}$ and is numeralwise defined thus is a deduction of A_E. So :

- S has graph defined (and is func. rep.) by $v_2 = Sv_1$
- $+$ has graph defined by $v_3 = v_1 + v_2$
- $\cdot$ has graph defined by $v_3 = v_1 \cdot v_2$
- E has graph defined by $v_3 = v_1 E v_2$
- I_i^m has graph defined by $v_{m+1} = v_i$
- $\chi_<$ has graph defined by $(v_1 < v_2 \wedge v_3 = \overline{1}) \vee (\neg(v_1 < v_2) \wedge v_3 = 0)$ which is numeralwise determined by Theorem 12.
- $\chi_=$ has graph defined by $(v_1 = v_2 \wedge v_3 = \overline{1}) \vee (\neg(v_1 = v_2) \wedge v_3 = 0)$ which is numeralwise defined by Theorem 12.

2. Let's prove the statement in the case $m = 1, n = 2$, i.e. we consider $f(a) = g(h_1(a), h_2(a))$
   By hypothesis, there are formulae $\psi, \theta_1, \theta_2 \in Fml_{\mathcal{L}_\mathbb{N}}$ func. rep. respectively g, h_1, h_2.
   Now let ϕ be $\forall y_1 \forall y_2(\theta_1(v_1, y_1) \to (\theta_2(v_1, y_2) \to \psi(y_1, y_2, v_2))$
   Let $a \in \mathbb{N}$:

$A_E, \phi(\overline{a}, v_2) \vdash \theta_1(\overline{a}, y_1) \to (\theta_2(\overline{a}, y_2) \to \psi(y_1, y_2, v_2))$ by Ax(H4)

$A_E, \phi(\overline{a}, v_2) \vdash y_1 = \overline{h_1(a)} \to (y_2 = \overline{h_2(a)} \to \psi(y_1, y_2, v_2))$ by def of θ_1, θ_2

$A_E, \phi(\overline{a}, v_2) \vdash y_1 = \overline{h_1(a)} \to (y_2 = \overline{h_2(a)} \to \psi(\overline{h_1(a)}, \overline{h_2(a)}, v_2))$ by Ax(H8)

$A_E, \phi(\overline{a}, v_2) \vdash y_1 = \overline{h_1(a)} \to (y_2 = \overline{h_2(a)} \to v_2 = \overline{g(h_1(a), h_2(a))})$ by def of ψ

$A_E, \phi(\overline{a}, v_2) \vdash v_2 = \overline{g(h_1(a), h_2(a))}$ by Gen, Ax(H4) and MP

$A_E \vdash \phi(\overline{a}, v_2) \to v_2 = \overline{g(h_1(a), h_2(a))}$ by Ded. Thm

$$
\begin{aligned}
A_E, v_2 = & \overline{f(a)}, \theta_1(\overline{a}, y_1), \theta_2(\overline{a}, y_2) & \\
& \vdash \psi(\overline{h_1(a)}, \overline{h_2(a)}, v_2) & \text{by Ax(H4) + def of } \psi \\
\text{——"——} & \vdash y_1 = \overline{h_1(a)} & \text{by Ax(H4) + def of } \theta_1 \\
\text{——"——} & \vdash y_2 = \overline{h_2(a)} & \text{by Ax(H4) + def of } \theta_2 \\
\text{——"——} & \vdash y_1 = \overline{h_1(a)} \to (y_2 = \overline{h_2(a)} \to \psi(\overline{h_1(a)}, \overline{h_2(a)}, v_2)) & \text{by Mon. + Ded. Thm} \\
\text{——"——} & \vdash y_1 = \overline{h_1(a)} \to (y_2 = \overline{h_2(a)} \to \psi(y_1, y_2, v_2)) & \text{by Ax(H8)} \\
\text{——"——} & \vdash \psi(y_1, y_2, v_2)) & \text{by MP} \\
A_E, v_2 = \overline{f(a)} & \vdash \theta_1(\overline{a}, y_1) \to (\theta_2(\overline{a}, y_2) \to \psi(y_1, y_2, v_2)) & \text{by Ded. Thm} \\
A_E & \vdash v_2 = \overline{f(a)} \to \phi(\overline{a}, v_2) & \text{by Gen and Ded. Thm}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
A_E &\vdash \phi(\overline{a}, v_2) \leftrightarrow v_2 = \overline{f(a)} & \text{by Right } \wedge\text{-Intro} \\
A_E &\vdash \forall v_2(\phi(\overline{a}, v_2) \leftrightarrow v_2 = \overline{f(a)}) & \text{by Gen}
\end{aligned}
$$

which means that ϕ functionally represents f as desired.
The general case is exactly the same proof with more notations.

3. Let's prove the statement in the case $m = 1$, i.e consider $f(a) = \mu b[g(a, b) = 0]$.
Let $\psi \in Fml_{\mathcal{L}_\mathbb{N}}$ functionally representing g.
Let ϕ be $\psi(v_1, v_2, 0) \wedge \forall y(y < v_2 \to \neg\psi(v_1, y, 0))$ Then ϕ defines the graph R_f of f and is numeralwise determined by A_E by Theorem 12 since

$$\phi \text{ functionally rep. } f \overset{\text{Theorem 21}}{\Rightarrow} \phi \text{ rep. } R_f \overset{\text{Theorem 7}}{\Rightarrow} \phi \text{ is numeralwise det.}$$

Thus we have our conclusion.
The general case is the same proof with more notations.

□

Theorem 25. Here is a repertoire of representable functions and relations. Capital letters will denote relation and non-capitals letter will denote functions.

(1) R is representable $\Leftrightarrow \chi_R$ is representable

(2) $R \subseteq \mathbb{R}^m$, $f_1, \ldots, f_m$ are representable $\Rightarrow \{\vec{a} \mid \langle f_1(\vec{a}), \ldots, f_m(\vec{a})\rangle \in R\}$ is representable

(3) $R \subseteq \mathbb{R}^{m+1}$ representable $\Rightarrow$ so are

$$
\begin{aligned}
P &= \{\langle a_1, \ldots, a_m, b\rangle \mid \text{for some } c < b,\ \langle a_1, \ldots, a_m, c\rangle \in R\} \\
P' &= \{\langle a_1, \ldots, a_m, b\rangle \mid \text{for all } c < b,\ \langle a_1, \ldots, a_m, c\rangle \in R\} \\
Q &= \{\langle a_1, \ldots, a_m, b\rangle \mid \text{for some } c \leqslant b,\ \langle a_1, \ldots, a_m, c\rangle \in R\} \\
Q' &= \{\langle a_1, \ldots, a_m, b\rangle \mid \text{for all } c \leqslant b,\ \langle a_1, \ldots, a_m, c\rangle \in R\}
\end{aligned}
$$

(4) The divisibility relation $R = \{\langle a, b\rangle \mid a \mid b \text{ in } \mathbb{N}\}$ is representable

(5) The set of primes $P = \{p \mid p \text{ is a prime}\}$ is representable

(6) The set of adjacent primes $R = \{\langle a, b\rangle \mid \text{a,b are primes and there is no prime c} : a < c < b\}$ is representable

(7) The function $f(n) = p_n$, where p_n is the $(n+1)^{\text{st}}$ prime number, is representable.

(8) The *encoding* functions $\ll a_0, \ldots, a_m \gg_m := p_0^{a_0+1} \ldots p_m^{a_m+1} = \prod_{i \leqslant m} p_i^{a_i+1}$ are representable. For simplicity we'll drop the subscript and write $\ll \ldots \gg$ and by convention $\ll\gg = 1$

(9) There is a representable *decoding* function $(\cdot)_b$ for each $b \in \mathbb{N}$ s.t when $b \leqslant m$: $(\ll a_0, \ldots, a_m \gg)_b = a_b$

Corollary 26. From part (9), we have that with a representable relation R s.t for every $\vec{a}$ there is some n s.t $\langle a, n\rangle \in R$. Then $f(a) :=$ the least n such that $\langle \vec{a}, n\rangle \in R$, is representable.

Definition 27. A number $b \in \mathbb{N}$ is a *sequence number* if $b = \ll a_0, \ldots, a_m \gg$, some $m, a_0, \ldots, a_m$

(10) Then the set *SeqNumb* of sequence number is representable.

(11) There is a representable *length* function lh s.t $lh(\ll a_0, \ldots, a_m \gg) = m$

(12) There are representable *restriction* functions $\cdot|_b$ for $b \in \mathbb{N}$ s.t $\ll a_0, \ldots, a_m \gg |_b = \ll a_0, \ldots, a_{b-1} \gg$

(13) Take a $(k+2)$-ary function g, then there exists a unique function f s.t

$$f(a, \vec{b}) = g(\tilde{f}(a, \vec{b}), a, \vec{b})$$
$$\text{where } \tilde{f}(a, \vec{b}) := \ll f(0, \vec{b}), \ldots, f(a-1, \vec{b}) \gg$$

This comes from the recursion, such as seen in a Set Theory unit.
But then if g is representable, so is f.
In a particular case, if g, h are representable functions and

$$f(0, \vec{b}) = g(\vec{b})$$
$$f(a+1, \vec{b}) = h(f(a, \vec{b}), a, \vec{b})$$

Then f is representable.

(14) f representable $\Rightarrow g(a, \vec{b}) = \prod_{i<a} f(i, \vec{b})$ is representable

(15) The *concatenation* $a * b = \prod_{i<lh(b)} p_{i+lh(a)}^{(b)_i+1}$ is representable and is called so since $\ll a_0, \ldots, a_m \gg * \ll b_0, \ldots, b_m \gg = \ll a_0, \ldots, a_m, b_0, \ldots, b_n \gg$

(16) For a representable function f, the *concatenation* of f, $\bigstar_{i<a} f(i) := f(0) * \ldots * f(a-1)$ is representable

Proof. (1) ($\Rightarrow$) Let $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ represent $R \subseteq \mathbb{R}^m$ with $FV(\phi) \subseteq \{v_1, \ldots, v_m\}$. Let θ be $(\phi \wedge v_{m+1} = \overline{1}) \vee (\neg\phi \wedge v_{m+1} = 0)$. Take $a_1, \ldots, a_m, b \in \mathbb{N}$ arbitrary :

$$\begin{aligned}
\langle a_1, \ldots, a_m, b\rangle \in R_{\chi_R} &\Rightarrow b = \chi_R(a_1, \ldots, a_m) \\
&\Rightarrow (\langle a_1, \ldots, a_m\rangle \in R \text{ and } b = 1) \text{ or} \\
&\quad\;\, (\langle a_1, \ldots, a_m\rangle \in R \text{ and } b = 0) \\
&\Rightarrow (A_E \vdash \overline{b} = \overline{1} \text{ and } A_E \vdash \phi(\overline{a_1}, \ldots, \overline{a_m})) \text{ or} && \text{by Hypothesis} \\
&\quad\;\, (A_E \vdash \neg\phi(\overline{a_1}, \ldots, \overline{a_m} \text{ and } A_E \vdash \overline{b} = 0) \\
&\Rightarrow A_E \vdash (\phi(\overline{a_1}, \ldots, \overline{a_m}) \wedge \overline{b} = \overline{1}) \vee && \text{by Right } \wedge, \vee\text{-Intro} \\
&\qquad\quad (\neg\phi(\overline{a_1}, \ldots, \overline{a_m}) \wedge \overline{b} = 0) \\
&\Rightarrow A_E \vdash \theta(\overline{a_1}, \ldots, \overline{a_m}, \overline{b}) \\
\langle a_1, \ldots, a_m, b\rangle \notin R_{\chi_R} &\Rightarrow A_E \vdash \neg\theta(\overline{a_1}, \ldots, \overline{a_m}, \overline{b}) && \text{Similarly}
\end{aligned}$$

So R_{χ_R} is representable, therefore by Theorem 22 χ_R is functionally representable.
($\Leftarrow$) Let $\psi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ functionally represent χ_R, with $FV(\psi) \subseteq \{v_1, \ldots, v_m\}$. Then ψ represents R_{χ_R} by Theorem 21, is numeralwise determined and defines R_{χ_R} by Theorem 7. So for arbitrary $\mathfrak{M}, s$:

$$\begin{aligned}
\mathfrak{M}, s \models \psi(v_1, \ldots, v_m, 0) &\Leftrightarrow \mathfrak{M}, s\tfrac{0}{v_{m+1}} \models \psi(v_1, \ldots, v_m, v_{m+1}) && \text{by Substitution Lemma} \\
&\Leftrightarrow \langle s\tfrac{0}{v_{m+1}}(v_1), \ldots, s\tfrac{0}{v_{m+1}}(v_{m+1})\rangle \in R_{\chi_R} && \text{since } \psi \text{ defines } R_{\chi_R} \\
&\Leftrightarrow \langle s(v_1), \ldots, s(v_m), 0\rangle \in R_{\chi_R} \\
&\Leftrightarrow \langle s(v_1), \ldots, s(v_m) \in R
\end{aligned}$$

So $\psi(v_1, \ldots, v_m, 0)$ defines R. And we have shown in the proof of Theorem 12 (c) that ψ is numeralwise determined $\Leftrightarrow$ $\psi(v_1, \ldots, v_m, 0)$ is numeralwise determined. Therefore by Theorem 7, $\psi(v_1, \ldots, v_m, 0)$ represents R.

(2) Let $P = \{\vec{a} \mid \langle f_1(\vec{a}), \ldots, f_m(\vec{a})\rangle \in R\}$. It's characteristic function is

$$\chi_P(\vec{a}) = \chi_R(f_1(\vec{a}), \ldots, f_m(\vec{a}))$$

which is functionally representable by Theorem 24 as a composition of functionally representable functions. So by (1), P is representable.

(3) Let $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ with $FV(\phi) \subseteq \{v_1, \ldots, v_{m+1}\}$ represent R. So by Theorem 7 ϕ is numeralwise determined and defines R. Let ψ be $\exists x(x < v_{m+1} \wedge \phi(v_1, \ldots, v_m, x))$
By Theorem 12 (c) : ϕ numeralwise determined $\Rightarrow$ ψ numeralwise determined
For any $\mathbb{M}, s$:

$$\begin{aligned}
\mathfrak{M}, s \models \psi &\Leftrightarrow \text{for some } k \in \mathbb{N} : k < s(v_{m+1}) \text{ and } \mathfrak{M}, s\tfrac{k}{x} \models \phi(v_1, \ldots, v_m, x) && \text{by def of } \psi \\
&\Leftrightarrow \text{for some } k < s(v_{m+1}) : \mathfrak{M}, s\tfrac{k}{v_{m+1}} \vdash \phi && \text{by Subst Lem} \\
&\Leftrightarrow \text{for some } k < s(v_{m+1}) : \langle s(v_1), \ldots, s(v_m), k\rangle \in R && \text{since } \phi \text{ defines } R \\
&\Leftrightarrow \langle s(v_1), \ldots, s(v_m), s(v_{m+1})\rangle \in P && \text{by def of } P
\end{aligned}$$

So ψ defines P, and therefore by Theorem 7, ψ represents P.

But now,

$$\begin{aligned}\langle a_1,\ldots,a_m,b\rangle\in Q &\Leftrightarrow \langle a_1,\ldots,a_m,c\rangle\in Q \text{ some } c\leqslant b\\ &\Leftrightarrow \langle a_1,\ldots,a_m,c\rangle\in Q \text{ some } c<b+1\\ &\Leftrightarrow \langle a_1,\ldots,a_m,b+1\rangle\in P\end{aligned}$$

So $Q = \{\langle a_1,\ldots,a_m,b\rangle \mid \langle I_1^{m+1}(\langle\vec{a},b\rangle),\ldots,I_m^{m+1}(\langle\vec{a},b\rangle),S(I_{m+1}^{m+1}(\langle\vec{a},b\rangle))\rangle\in P\}$ is representable by (2). And similarly P', Q' are representable.

(4) Notice that $a\mid b \Leftrightarrow b=ac$ some $c\leqslant b$.

$$\overset{(2)}{\Rightarrow} R_1=\{\langle a,b,c\rangle \mid \langle a\cdot c,b\rangle\in R_=\} \text{ is representable, where } R_= \text{ is the graph of } =$$

$$\overset{(3)}{\Rightarrow} R_2=\{\langle a,b,d\rangle \mid \text{for some } c\leqslant d:\langle a,b,c\rangle\in R_1\} \text{ is representable}$$

$$\overset{(2)}{\Rightarrow} R=\{\langle a,b\rangle \mid \langle a,b,b\rangle\in R_2\} \text{ is representable}$$

(5) Note that p is a prime $\Leftrightarrow$ for $x\leqslant p: x\mid p$ iff $x\in\{1,p\}$. By (4), let $\phi\in Fml_{\mathcal{L}_\mathbb{N}}$ represents the divisibility relation R. So ϕ also defines R and is numeralwise determined by Theorem 7. Let ψ be $\overline{1}<y\wedge(\phi(x,y)\rightarrow x=\overline{1}\vee x=y)$ which is also numeralwise determined by Theorem 12. Let

$$P_1=\{\langle a,c\rangle \mid 1<a\wedge(c\mid a\rightarrow c=1\vee c=a)\}$$

So

$$\begin{aligned}\mathfrak{N}\models\psi(\overline{p},\overline{n}) &\Leftrightarrow 1<p \text{ and if } \mathfrak{N}\models\phi(\overline{n},\overline{p}) \text{ then } \overline{n}=\overline{1}\vee\overline{n}=\overline{p}\\ &\Leftrightarrow 1<p \text{ and if } n\mid p \text{ then } n=1 \text{ or } n=p\\ &\Leftrightarrow \langle p,n\rangle\in P_1\end{aligned}$$

So ψ defines P_1 and thus represents it.

$$\Rightarrow P_1 \text{ is representable}$$

$$\overset{(3)}{\Rightarrow} P_2=\{\langle a,b\rangle \mid \text{for some } c\leqslant b:\langle a,c\rangle\in P_1\} \text{ is representable}$$

$$\overset{(2)}{\Rightarrow} P=\{p \mid \langle p,p\rangle\in P_2\} \text{ is representable}$$

(6) By (5), let $\phi\in Fml_{\mathcal{L}_\mathbb{N}}$ represents the set of primes. So ϕ also defines it and is numeralwise determined by Theorem 7. Let ψ be

$$\phi(x)\wedge\phi(y)\wedge\forall z(z<y\rightarrow\neg(x<y\wedge\phi(z)))$$

Then ψ is numeralwise determined by Theorem 12 and

$$\begin{aligned}\mathfrak{N}\models\psi(\overline{a},\overline{b}) &\Leftrightarrow \mathfrak{N}\models\phi(\overline{a}) \text{ and } \mathfrak{N}\models\phi(\overline{b}) \text{ and for all c : if } c<b\\ &\quad\ \text{then we can't have } a<b \text{ and } \mathfrak{N}\models\phi(\overline{c})\\ &\Leftrightarrow \text{a,b are primes and there is no prime c} : a<c<b\\ &\Leftrightarrow \langle a,b\rangle\in R\end{aligned}$$

So ψ defines R and is numeralwise determined, thus represents R by Theorem 7.

(7) First we shall prove the following criteria : $b = p_n \Leftrightarrow b$ is prime and there exists $c \leqslant b^{n^2}$ s.t

(i) $2 \nmid c$

(ii) if $q < b$ and $r \leqslant b$ are adjacent prime numbers then for all $j < c$: $q^j \mid c \Leftrightarrow r^{j+1} \mid c$

(iii) $b^n \mid c$ and $b^{n+1} \nmid c$

($\Rightarrow$) If $b = p_n$, let $c = 2^0 \cdot 3^1 \cdot 5^2 \dots p_n^n = \prod_{i \leqslant n} p_i^i$. Observe that c is odd, if $i < n$,$j < c$ then

$$p_i^j \mid c \Leftrightarrow j \leqslant i \Leftrightarrow j+1 \leqslant i+1 \Leftrightarrow p_{j+1}^{i+1} \mid c$$

and $p_n^n \mid c$ but $p_n^{n+1} \nmid c$. So c satisfy (i)-(iii) ($\Leftarrow$) Suppose b is prime and there is some $c \leqslant b^{a^2}$ satisfying (i)-(iii). So $b = p_m$ some $m \in \mathbb{N}$.

- If $c = 1$: (ii) $\Rightarrow$ as $p_o^o \mid c$ but $p_1^1 \nmid c$, $p1 = 3 > b$ so $b = 2, m = 0$ and (iii) $\Rightarrow n = 0$, therefore $m = n$
- Otherwise, by the Fundamental Theorem of Arithmetic, we can write c as $c = p_o^{a_0} \dots p_q^{a_q}$ some $q, a_o, \dots, a_q \in \mathbb{N}$
(i) $\Rightarrow a_0 = 0$
(ii) $\Rightarrow$ for $k < m, j < c$: $p_k^k \mid c \Leftrightarrow p_{k+1}^{k+1} \mid c$. So as $a_0 = 0$, we must have $p_1^1 \mid c$ but $p_1^2 \nmid c$, thus $a_1 = 1$ and inductively $a_l = l$ for $l \leqslant m$.
(iii) $\Rightarrow b^n \mid c$ and $b^{n+1} \nmid c$ so $p_m \mid c$ and $a_m = n$. But as $a_m = m$, we have $m = n$ and $b = p_m = p_n$
And this criteria makes the representation clear : let ϕ represent the set of primes, ψ represent the set of adjacent primes and θ represent the divisibility relation. Then the graph of f is

$$R_f = \{\langle a, b\rangle \mid \langle a, b, b\rangle \in \{\langle a, b, d\rangle \mid \text{for some } c \leqslant d : \langle a, b, c\rangle \in R\}\}$$

where R is represented by

$$\neg\theta(\overline{2}, z) \wedge \forall u(v < Sb \to \forall v(\psi(u, v) \to \forall w(w < c \to (\theta(uEw, z) \leftrightarrow \theta(vESw, z)))) \\ \wedge \theta(yEx, z) \wedge \neg\theta(yESx)$$

(8) Let f be the function enumerating primes, then $\ll \dots \gg_m = \prod_{i \leqslant m} f(i)Ea_i$ is representable as a composition of representable functions by Theorem 24.

(9) Define $(\cdot)_b : a \mapsto \mu n[a = 0 \text{ or } p_b^{n+2} \nmid a] = min\{n \mid a = 0 \text{ or } p_b^{n+2} \nmid a\}$. So observe that, as desired, $(a = \ll a_0, \dots, a_m \gg)_b = a_b$ as $a \neq 0$ and $p_b^{a_b+1} \mid a$ but $p_b^{a_b+2} \nmid a$.
Let $R = \{\langle a, b, n\rangle \mid a = 0 \text{ or } p_b^{n+2} \nmid a\}$, represented by θ which is

$$x = 0 \vee \forall u(u \leqslant x \to (\psi(y, u) \to \neg\phi(uESSz, x)))$$

where ϕ represents the divisibility relation and ψ represents the graph of the enumerating prime function.

$$\Rightarrow \text{ its complement } R^c \text{ is represented by } \neg\theta$$
$$\overset{(1)}{\Rightarrow} \chi_{R^c} \text{ is representable}$$
$$\overset{\text{Theorem 24}}{\Rightarrow} \mu n[\chi_{R^c}(a, b, n) = 0] \text{ is representable}$$

and $\mu n[\chi_{R^c}(a, b, n) = 0] = \mu n[\langle a, b, n\rangle \notin R^c] = \mu n[\langle a, b, n\rangle \in R] = (a)_b$

(10) The set of sequence numbers can be written as

$$R = \{a \mid \text{for all } p \leqslant a : (\text{If } p \mid a \text{ and } p \text{ prime then for all } q \leqslant p : (\text{If } q \text{ prime then } q \mid a))\}$$

which is representable by (3),(4) and (5).

(11) We define $lh(a) = \mu n[a = 0 \text{ or } p_n \nmid a]$. This function does what we want and is representable by Corollary 26

(12) We defined $a|_b = \mu n[a = 0 \text{ or for any } i < b, j < a : p_i^j \mid a \Rightarrow p_i^j \mid n]$. This function does what we want and is representable by Corollary 26

(13) Observe that

$$\begin{aligned}\tilde{f}(a, \vec{b}) &= \ll f(0, \vec{b}), \ldots, f(a-1, \vec{b}) \gg \\ &= \mu n[n \text{ is a sequence number} \wedge lh(n) = a - 1 \wedge \forall i < a((n)_i = g(n|_i, i, \vec{b}))] \\ &= \mu n[\langle a, \vec{b}, n\rangle \in R]\end{aligned}$$

where

$$\begin{aligned}R = \{\langle a, \vec{b}, n \mid n \in SeqNumb\} &\cap \{\langle a, \vec{b}, n\rangle \mid \langle S(lh(n)), a\rangle \in R_=\} \\ &\cap \{\langle a, \vec{b}, n\rangle \mid \text{for all } i < a : \langle a, \vec{b}, i\rangle \in P\}\end{aligned}$$

$$P = \{\langle a, \vec{b}, i\rangle \mid \langle (n)_i, g(n|_i, i, \vec{b})\rangle \in R_=\}$$

are both representable by (2),(3),(10),(11) and (12). Therefore $\tilde{f}$ is representable by Corollary 26 and so is f by Theorem 24 as a composition of representable functions.
In the particular case, observe that $f(a, \vec{b}) = k(\tilde{f}(a, \vec{b}), a, \vec{b})$ where

$$k(x, a, \vec{b}) := \begin{cases} g(\vec{b}) & \text{if } x = 0 \\ h((x)_m, m, \vec{b}) & \text{if } a = m + 1 \end{cases}$$

is representable since its graph is too (by (2),(3), and (9)) :

$$\begin{aligned}R_k = &\{\langle x, a, \vec{b}, y\rangle \mid x = 0 \wedge y = g(b)\} \\ &\cup \{\langle x, a, b, y\rangle \mid \text{for some } m < a : a = Sm \wedge y = h((x)_m, m, b)\}\end{aligned}$$

(14) We have

$$g(0, \vec{b}) = 1 \text{ (empty product)}$$
$$g(a + 1, \vec{b}) = g(a, \vec{b}) \cdot f(a, \vec{b})$$

So we get our conclusion immediately with (13).

(15) This is clear by applying (14) with $f(i, a, b) = g(i + lh(a))ES((b)_i)$ which is representable.

(16) We have

$$\mathop{\star}_{i<0} f(i, \vec{b}) = \ll\gg = 1$$
$$\mathop{\star}_{i<a+1} f(i, \vec{b}) = \mathop{\star}_{i<a} f(i, \vec{b}) * f(a, \vec{b})$$

So the argument ends just as in the proof of (14).

□

4 Gödel Numbering & Key representable sets and functions

By making correspondences between terms, formulae and deductions, and natural numbers, we'll be able to express new results and assertions. For that sake, consider the following Gödel Numbering :

Definition 28. • Let $\# : \mathcal{A}_{\mathcal{L}_\mathbb{N}} \to \mathbb{N}$ be the injection given by

Non-logical Symbols	Logical Symbols
$0 \mapsto 0$	$(\mapsto 1$
$S \mapsto 2$	$) \mapsto 3$
$< \mapsto 4$	$, \mapsto 5$
$+ \mapsto 6$	$= \mapsto 7$
$\cdot \mapsto 8$	$\neg \mapsto 9$
$E \mapsto 10$	$\to \mapsto 11$
	$\forall \mapsto 13$
	$v_i \mapsto 15 + 2i$

- Let $\#^* : \mathcal{A}^*_{\mathcal{L}_\mathbb{N}} \to \mathbb{N} : \langle x_0, \ldots, x_k \rangle \mapsto \ll \#x_o, \ldots, \#x_k \gg$ which is an injection.
- Let $\#^{**} : \mathcal{A}^{**}_{\mathcal{L}_\mathbb{N}} \to \mathbb{N} : \langle y_0, \ldots, y_k \rangle \mapsto \ll \#^*y_0, \ldots, \#^*y_k \gg$ which is an injection.

And when the context of if we have to deal with a symbol, term, formulae or deduction is clear, we'll omit the asterisk(s) and just write # for the concerned injection. And for a set $X \subseteq \mathcal{A}_{\mathcal{L}_\mathbb{N}}$, $\mathcal{A}^*_{\mathcal{L}_\mathbb{N}}$ or $\mathcal{A}^{**}_{\mathcal{L}_\mathbb{N}}$ of expressions, we'll denote $\#X := \{\#x \mid x \in X\}$

Remark. When having a Gödel Numbering, something to care about if the following relations are representable :

$$R_{PredS} := \{\langle a, b\rangle \mid \text{for some b-ary predicate symbol P, } a = \#P\}$$
$$R_{FuncS} := \{\langle a, b\rangle \mid \text{for some b-ary function symbol f, } a = \#f\}$$

In our situation, it's the case since both are finite :

$$R_{PredS} := \{\langle 4, 2\rangle\}$$
$$R_{FuncS} := \{\langle 2, 1\rangle, \langle 6, 2\rangle, \langle 8, 2\rangle, \langle 10, 2\rangle, (\text{and eventually } \langle 0, 0\rangle)\}$$

Where $\langle 0, 0\rangle \in R_{FuncS}$ whether or not we consider constant symbols as 0-ary function symbols.

From now on, we're going to helps us a little bit by making use of maths abbreviation inside English sentences, as distinction between English and formal language should be clear.

Theorem 29. Here is a repertoire of representable sets and functions :

(1) $\#Var$ is representable

(2) $\#Tm_{\mathcal{L}_\mathbb{N}}$ is representable

(3) $\#AtFml$, the set of atomic formulae, is representable

(4) $\#Fml$ is representable

(5) There is a representable substitution function Sub :

$$Sub(a,b,c) := \begin{cases} \#[\alpha]\frac{t}{x}, & \text{if } a = \#\alpha, b = \#x, c = \#t, \text{ some } \alpha \in Fml_{\mathcal{L}_{\mathbb{N}}}, x \in Var, t \in Tm_{\mathcal{L}_{\mathbb{N}}} \\ 0, & \text{o.w} \end{cases}$$

(6) The function $f : n \mapsto \#\overline{n}$ is representable

(7) There exists a representable relation Fr where :

$$\begin{aligned} \langle a,b \rangle \in Fr \iff & a = \#\alpha, \text{ some } \alpha \in Fml_{\mathcal{L}_{\mathbb{N}}} \\ & \wedge b = \#x, \text{ some } x \in Var \\ & \wedge x \in FV(\alpha) \end{aligned}$$

(8) $\#Sent_{\mathcal{L}_{\mathbb{N}}}$ is representable

(9) There exists a representable relation $Subt$ where :

$$\begin{aligned} \langle a,b,c \rangle \in Subt \iff & a = \#\alpha, \text{ some } \alpha \in Fml_{\mathcal{L}_{\mathbb{N}}} \cup Tm_{\mathcal{L}_{\mathbb{N}}} \\ & \wedge b = \#x, \text{ some } x \in Var \\ & \wedge c = \#t, \text{ some } t \in Tm\mathcal{L}_{\mathbb{N}} \\ & \wedge \text{ t is substitutable in } \alpha \text{ for } x \end{aligned}$$

(10) The generalisation relation Gen is representable :

$$\begin{aligned} \langle a,b \rangle \in Gen \iff & a = \#\alpha, \text{ some } \alpha \in Fml_{\mathcal{L}_{\mathbb{N}}} \\ & \wedge b = \#\forall x_1 \dots \forall x_m \alpha, \text{ some } x_1, \dots, x_m \in Var \end{aligned}$$

(11) $\#H1, \dots, \#H8$ and $\#\Lambda$ are representable.

(12) For $A \subseteq Fml_{\mathcal{L}_{\mathbb{N}}}$: $\#A$ representable $\Rightarrow \#Ded_A$ is recursive. Where

$$Ded_A := \{\mathcal{D} \in \mathcal{A}^{**}_{\mathcal{L}_{\mathbb{N}}} \mid \mathcal{D} \text{ is a deduction from } A\}$$

(13) For a relation R : R recursive $\Rightarrow R$ representable in CnA_E

Theorem 30. For a relation R : R is recursive $\iff$ R is representable in CnA_E

Corollary 31. For a relation R : R is recursive $\Rightarrow R$ is definable in $\mathfrak{N}$

(14) For $A \subseteq Fml_{\mathcal{L}_{\mathbb{N}}}$: $\#A$ recursive and CnA a complete theory $\Rightarrow \#$CnA is recursive.

(15) For $\Gamma \subseteq Sent_{\mathcal{L}_{\mathbb{N}}}$: $\#\Gamma$ recursive $\Rightarrow$ the proof relation $Proof_\Gamma$ is recursive, where

$$\langle x,y \rangle \in Proof_\Gamma :\Leftrightarrow y = \#\phi, \text{ some } \phi \in Fml_{\mathcal{L}_{\mathbb{N}}} \wedge x = \#\mathcal{D} \text{ a deduction of } \phi \text{ from } \Gamma$$

Proof. (1) We can write it as $\#Var = \{a \mid \text{for some } b < a : \langle a, 15+2b \rangle \in R_=\}$ which is representable by Theorem 25,(2) and (3)

(2) Observe that

$$x \in Tm_{\mathcal{L}_\mathbb{N}} \Leftrightarrow x \text{ is an atomic term}$$
$$\vee\ \exists t_0, \ldots, t_{m-1} \in Tm_{\mathcal{L}_\mathbb{N}} \exists \text{ m-ary function symbol } f : x = f(t_0, \ldots, t_{m-1})$$

For this second case, we will express the existence of a sequence number i, encoding the information $i = \ll \#t_0, \ldots, \#t_{m-1} \gg$. So, with $n = \#x$, an effective bound is :

$$\begin{aligned}
i &= 2^{\#t_0+1} \ldots p_{m-1}^{\#t_{m-1}+1} \\
&\leqslant 2^n \ldots p_{m-1}^n \text{ since } \#t_l + 1 \leqslant n,\ \forall\ 0 \leqslant l \leqslant m-1 \text{ as } t_l \text{ is a subseq. of x in terms of symbols} \\
&< 2^n \ldots p_{lh(n)-1}^n \text{ since } m = lh(i) < lh(n) \\
&\leqslant n^n \ldots p_{lh(n)-1}^n \text{ since } p_{lh(n)-1} \leqslant 2^{(n)_0+1} \ldots p_{lh(n)-1}^{(n)_{lh(n)-1}+1=n} \\
&= n^{nlh(n)}
\end{aligned}$$

Therefore we have for $f = \chi_{\#Tm_{\mathcal{L}_\mathbb{N}}}$:

$$f(n) = \begin{cases} 1 & \text{if } n \in \#Var \cup \{0\} \vee \exists i < n^{nlh(n)}, \exists k < n : \\ & (i \in SeqNumb \wedge \text{ for all } j < lh(i) : f((i)_j) = 1 \\ & \qquad \wedge \langle k, lh(i) \rangle \in R_{FuncS} \\ & \qquad \wedge\ n = \ll k, \#(\gg * \bigstar_{j<lh(i)-1} ((i)_j * \ll \#, \gg) * \ll (i)_{lh(i)}, \#) \gg) \\ 0 & \text{otherwise} \end{cases}$$

Notice that the last line of the '1 scenario' is somehow complicated, because of our notation for a term : $f(t_1, \ldots, t_m)$. Whereas if we were using the lighter notation $ft_1 \ldots t_m$ it would be a bit easier.
As observed, f has a recursive description, so we'll write it as
$f(n) = g(\tilde{f}(n), n)$ and use the primitive recursion of Theorem 25.(13). This is done as follow :

$$g(m, n) = \begin{cases} 1 & \text{if } n \in \#Var \cup \{0\} \vee \exists i < n^{nlh(n)}, \exists k < n : \\ & (i \in SeqNumb \wedge \text{ for all } j < lh(i) : (m)_{(i)_j} = 1 \\ & \qquad \wedge \langle k, lh(i) \rangle \in R_{FuncS} \\ & \qquad \wedge\ n = \ll k, \#(\gg * \bigstar_{j<lh(i)-1} ((i)_j * \ll \#, \gg) * \ll (i)_{lh(i)}, \#) \gg) \\ 0 & \text{otherwise} \end{cases}$$

We need to show that g is recursive. This is pretty obvious considering all the tools provided in Theorem 25, but let's make this one in detail :

$$
\begin{aligned}
R_g &= (R_1 \cup R_2) \cap R_3 \cup (R_1 \cup R_2)^c \cap R_4 \\
R_1 &= \{\langle m, n, x\rangle \mid n \in \#Var \cup \{0\}\} \\
R_2 &= \{\langle m, n, x\rangle \mid \langle m, n, x, nElh(n)\rangle \in S_1\} \\
R_3 &= \{\langle m, n, x\rangle \mid x = 1\} \\
R_4 &= \{\langle m, n, x\rangle \mid x = 0\} \\
S_1 &= \{\langle m, n, x, y\rangle \mid \text{for some } i < y : \langle m, n, x, i\rangle \in S_2\} \\
S_2 &= \{\langle m, n, x, i\rangle \mid \langle m, n, x, i, n\rangle \in S_3\} \\
S_3 &= \{\langle m, n, x, i, z\rangle \mid \text{for some } k < z : \langle m, n, x, i, k\rangle \in T_1 \cap T_2 \cap T_3 \cap T_4\} \\
T_1 &= \{\langle m, n, x, i, k\rangle \mid i \in SeqNumb\} \\
T_2 &= \{\langle m, n, x, i, k\rangle \mid \langle m, n, x, i, k, lh(i)\rangle \in U_1\} \\
U_1 &= \{\langle m, n, x, i, k, w\rangle \mid \text{for all } j < w :\ \langle m, n, x, i, k, j\rangle \in U_2\} \\
U_2 &= \{\langle m, n, x, i, k, j\rangle \mid (m)_{(i)_j} = 1\} \\
T_3 &= \{\langle m, n, x, i, k\rangle \mid \langle k, lh(i)\rangle \in R_{FuncS}\} \\
T_4 &= \{\langle m, n, x, i, k\rangle \mid n = \ll k, \#(\gg * \mathop{\star}_{j<lh(i)-1} ((i)_j * \ll \#, \gg) * \ll (i)_{lh(i)}, \#) \gg\}
\end{aligned}
$$

Each of those sets being representable $\Rightarrow$ g is representable $\Rightarrow$ f is representable $\Rightarrow$ $\#Tm_{\mathcal{L}_{\mathbb{N}}}$ is representable.

(3) Observe that

$$
\begin{aligned}
n \in \#AtFml \Leftrightarrow\ & n = \#P(t_0, \ldots, t_{m-1}) \text{ some m-ary pred. symbol } P \text{ and } t_0, \ldots, t_{m-1} \in Tm_{\mathcal{L}_{\mathbb{N}}} \\
\Leftrightarrow\ & \exists i < n^{nlh(n)}, \exists k < n[i \in SeqNumb \\
& \wedge \forall j < lh(i) : (i)_j \in \#Tm_{\mathcal{L}_{\mathbb{N}}} \\
& \wedge \langle k, lh(i)\rangle \in R_{PredS} \\
& \wedge n = \ll k, \#(\gg * \mathop{\star}_{j<lh(i)-1} ((i)_j * \ll \#, \gg) * \ll (i)_{lh(i)}, \#) \gg]
\end{aligned}
$$

And then proceed as in (2)

(4) Observe that for $f = \chi_{\#Fml_{\mathcal{L}_{\mathbb{N}}}}$:

$$
f(n) = \begin{cases}
1 & \text{if } n \in \#AtFml \\
 & \vee \text{ if } \exists i < n[f(i) = 1 \wedge n = \ll \#(, \#\neg \gg *i* \ll \#) \gg] \\
 & \vee \text{ if } \exists i, j < n[f(i) = 1 \wedge f(j) = 1 \wedge n = \ll \#(\gg *i* \ll \# \rightarrow\gg *j* \ll \#) \gg] \\
 & \vee \text{ if } \exists i, j < n[f(i) = 1 \wedge \chi_{\#Var}(j) = 1 \wedge n = \ll \#\forall \gg *j*i] \\
0 & \text{otherwise}
\end{cases}
$$

And then proceed as in (2)

(5) Observe that :

$$Sub(a,b,c) = \begin{cases} 1) & \text{If } a \in \#Var \wedge a = b \Rightarrow Sub(a,b,c) = c \\ 2) & \text{If } \exists i < a^{alh(a)} \exists k < a[i \in SeqNumb \\ & \qquad \wedge \forall j < lh(i) : (i)_j \in Tm_{\mathcal{L}_\mathbb{N}} \\ & \qquad \wedge \langle k, lh(i)\rangle \in R_{PredS} \\ & \qquad \wedge a = \ll k, \#(\gg * \underset{j<lh(i)-1}{\star} ((i)_j * \ll \#, \gg) * (i)_{lh(i)} * \ll \#) \gg] \\ & \Rightarrow Sub(a,b,c) = \ll k, \#(\gg * \underset{j<lh(i)-1}{\star} (Sub((i)_j, b, c) * \ll \#, \gg) \\ & \qquad\qquad * Sub((i)_{lh(i)}, b, c) * \ll \#) \gg \text{, with same i,k} \\ 3) & \text{If } \exists i < a[i \in \#Fml_{\mathcal{L}_\mathbb{N}} \wedge a = \ll \#(, \#\neg \gg *i* \ll \#) \gg] \\ & \Rightarrow Sub(a,b,c) = \ll \#(, \#\neg \gg *Sub(i,b,c)* \ll \#) \gg \text{, with same i} \\ 4) & \text{If } \exists i,j < a[i,j \in \#Fml_{\mathcal{L}_\mathbb{N}} \wedge a = \ll \#(\gg i* \ll \# \to\gg *j \ll \#) \gg] \\ & \Rightarrow Sub(a,b,c) = \ll \#(\gg Sub(i,b,c)* \ll \# \to\gg *Sub(j,b,c) \ll \#) \gg] \\ & \text{ with same i,j} \\ 5) & \text{If } \exists i,j < a[i \in \#Var \wedge i \neq b \wedge j \in \#Fml_{\mathcal{L}_\mathbb{N}} \wedge a = \ll \#\forall \gg *i*j] \\ & \Rightarrow Sub(a,b,c) = a = \ll \#\forall \gg *i * Sub(j,b,c)\text{, with same i,j} \\ 6) & \text{If (1)-(5) fail } \Rightarrow Sub(a,b,c) = 0 \end{cases}$$

(6) Observe that

$$f(0) = \ll \#0 \gg$$
$$f(n+1) = \ll \#S \gg *f(n)$$

We get our conclusion immediately from Theorem 25.(13)

(7) Observe that for $\alpha \in Fml_{\mathcal{L}_\mathbb{N}}, x \in Var$:

$$\langle \#\alpha, \#x \rangle \in Fr \iff x \in FV(\alpha) \iff [\alpha]\tfrac{0}{x} \neq \alpha$$

So $Fr = \{\langle a,b\rangle \mid Sub(a,b,\#\bar{0}) \neq a \wedge Sub(a,b,\#\bar{0}) \neq 0\}$ is representable.

(8) This comes from the fact that :

$$\#Sent_{\mathcal{L}_\mathbb{N}} = \{a \mid a \in \#Fml_{\mathcal{L}_\mathbb{N}} \wedge \forall b < a[b \in \#Var \to \langle a,b\rangle \notin Fr]\}$$

(9) Observe that for $f = \chi_{Subt}$:

$$f(a,b,c) = \begin{cases} 1 & \text{if } a \in \#AtFml_{\mathcal{L}_\mathbb{N}} \cup \#Tm_{\mathcal{L}_\mathbb{N}} \wedge b \in \#Var \wedge c \in \#Tm_{\mathcal{L}_\mathbb{N}} \\ & \vee \text{ if } \exists i < a(a = \ll \#(, \#\neg \gg *i* \ll \#) \gg \wedge f(i,b,c)) \\ & \vee \text{ if } \exists i,j < a(a = \ll \#(\gg *i* \ll \# \to\gg *j*\#) \gg \wedge f(i,b,c) \wedge f(j,b,c)) \\ & \vee \text{ if } \exists i,j < a(j \in \#Var \wedge a = \ll \#\forall \gg *j*i \\ & \qquad\qquad \wedge [\langle a,b\rangle \notin Fr \vee (\langle c,j\rangle \notin Fr \wedge f(i,b,c))] \\ 0 & \text{otherwise} \end{cases}$$

And then proceed as in (2).

(10) Observe that for $f = \chi_{\#Gen}$:

$$f(a,b) = \begin{cases} 1 & \text{if } a = b \\ & \vee \text{ if } \exists i,j[i \in \#Var \wedge b = \ll \#\forall \gg * i * j \wedge f(a,j) = 1] \\ 0 & \text{otherwise} \end{cases}$$

And then proceed as in (2).

(11) Observe that $\#H1 = \#\{\psi \in Fml_{\mathcal{L}_{\mathbb{N}}} \mid \psi = \phi \to (\theta \to \phi),\ \text{some } \phi, \theta \in Fml_{\mathcal{L}_{\mathbb{N}}}\}$, so

$$\begin{aligned} a \in \#H1 \iff & \exists i,j < a(i,j \in \#Fml_{\mathcal{L}_{\mathbb{N}}} \\ & \wedge a = \ll \#(\gg * i * \ll \# \to, \#(\gg * j * \ll \# \to \gg * i * \ll \#) \gg \end{aligned}$$

And then proceed as in (2). For $\#H2 - \#H8$ it's exactly the same story. Moreover :

$$a \in \#\Lambda \iff \exists b < a(b \in \#H1 \cup \ldots \cup \#H8 \wedge \langle a,b \rangle \in Gen)$$

So we've got our conclusion.

(12) Observe that

$$\begin{aligned} a \in \#\{\mathcal{D} \mid \mathcal{D} \text{ is a deduction from } A\} \iff & a \in SeqNumb \wedge 0 < lh(a) \wedge \forall i < lh(a) \\ & [(a)_i \in \#A \cup \#\Lambda \vee \exists j,k < i \\ & ((a)_j = \ll \#(\gg * (a)_k * \ll \# \to \gg * (a)_i * \ll \#) \gg)] \end{aligned}$$

(13) R recursive means that it is representable in some consistent finitely axiomatizable theory $\Rightarrow$ There exists $A \subseteq Sent_{\mathcal{L}_{\mathbb{N}}}$ consistent finite and there exists $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ representing R in CnA.
Say that R is an m-ary relation, then for each $a_1, \ldots, a_m \in \mathbb{N}$, as ϕ represents R, we have that ϕ is numeralwise determined by A, so there always exists a deduction $\mathcal{D}$ of either $\phi(\overline{a_1}, \ldots, \overline{a_m})$ or $\neg\phi(\overline{a_1}, \ldots, \overline{a_m})$. Therefore the following function is well-defined :

$$\begin{aligned} f(a_1, \ldots, a_m) = \mu n[& n \in \#\{\mathcal{D} \mid \mathcal{D} \text{ is a deduction from } A\} \\ & \wedge ((n)_{lh(n)} = \#\phi(\overline{a_1}, \ldots, \overline{a_m}) \vee (n)_{lh(n)} = \#\neg\phi(\overline{a_1}, \ldots, \overline{a_m}))] \end{aligned}$$

and is representable in CnA_E by Corollary 26. Therefore

$$R = \{\langle a_1, \ldots, a_m \rangle \mid f(\vec{a})_{lh(f(\vec{a}))} = \#\phi(\overline{a_1}, \ldots, \overline{a_m})\}$$

is representable since

$$\langle a_1, \ldots, a_m \rangle \mapsto \#\phi(a_1, \ldots, a_m) = \ll \#\phi, \#(\gg * \underset{j<m-1}{\star} (\ll \#\overline{a_j}, \#, \gg) * \ll \#\overline{a_m}, \#) \gg$$

is representable by (6)

(14) Observe that since CnA is complete, then the following function is well-defined :

$$\begin{aligned} f(a) = \mu n[& a \notin SeqNumb \vee (n \in \#DedA \\ & \wedge ((n)_{lh(n)} = a \vee (n)_{lh(n)} = \ll \#(,\#\neg \gg * a * \ll \#) \gg))] \end{aligned}$$

and is representable by Corollary 26. This function allows us to describe CnA :

$$\text{Cn}A = \{a \mid a \in \#Sent_{\mathcal{L}_{\mathbb{N}}} \wedge (f(a))_{f(a)} = a\}$$

which is representable.

(15) Observe that

$$Proof_\Gamma = \{\langle x, y\rangle \mid x \in \#Ded_\Gamma \wedge y \in \#Fml_{\mathcal{L}_\mathbb{N}} \wedge (x)_{lh(x)} = y\}$$

and thus it's a representable relation.

□

5 First Incompletness Theorem

Lemma 32 (Fixed Point Lemma)**.** For $\phi \in Fml_{\mathcal{L}_{\mathbb{N}}}$ with only one free variable $FV(\phi) \subseteq \{x\}$, then there exists $\sigma \in Sent_{\mathcal{L}_{\mathbb{N}}}$ s.t

$$A_E \vdash \phi(\overline{\#\sigma}) \leftrightarrow \sigma$$

Proof. By Theorem 29.(5) and (6) : $Sub(a, \#y, c)$ and $f : n \mapsto \#\overline{n}$ are functionally representable in CnA_E, thus so is $Sub(a, \#y, \overline{\#c})$, say by $\theta \in Fml_{\mathcal{L}_{\mathbb{N}}}$ with 3 free variables, w.l.o.g $FV(\theta) \subseteq \{v_1, v_2, x\}$ and take $y \in Var, y \neq x, v_1, v_2$.

$$\Rightarrow \text{For all } a, c \in \mathbb{N} : A_E \vdash \forall x(\theta(\overline{a}, \overline{c}, x) \leftrightarrow x = \overline{Sub(a, \#y, \overline{\#c})}$$
$$\Rightarrow \text{For all } \alpha \in Fml_{\mathcal{L}_{\mathbb{N}}}, c \in \mathbb{N} : A_E \vdash \forall x(\theta(\overline{\#\alpha}, \overline{c}, x) \leftrightarrow x = \overline{\#[\alpha]\tfrac{\overline{c}}{y}}) \quad (6)$$

Let

$$\rho := \forall x(\theta(y, y, x) \to \phi)$$
$$\sigma := \rho(\overline{\#\rho}) = \forall x(\theta(\overline{\#\rho}, \overline{\#\rho}, x) \to \phi)$$

To prove : $A_E \vdash \phi(\overline{\#\sigma}) \leftrightarrow \sigma$
($\to$) We have

$A_E, \sigma \vdash \sigma$	
$A_E, \sigma \vdash \theta(\overline{\#\rho}, \overline{\#\rho}, \overline{\#\sigma}) \to \phi(\overline{\#\sigma})$	By Right $\forall$-Elim
And $A_E, \sigma \vdash \forall x(\theta(\overline{\#\rho}, \overline{\#\rho}, x) \leftrightarrow x = \overline{\#\rho(\overline{\#\rho})})$	By (6) and Monicity
$A_E, \sigma \vdash \theta(\overline{\#\rho}, \overline{\#\rho}, \overline{\#\sigma}) \leftrightarrow \overline{\#\sigma} = \overline{\#\sigma}$	By Right $\forall$-Elim
$A_E, \sigma \vdash \overline{\#\sigma} = \overline{\#\sigma}$	By Ax(H7)
$A_E, \sigma \vdash \phi(\overline{\#\sigma})$	By MP twice
$A_E \vdash \sigma \to \phi(\overline{\#\sigma})$	By Ded Thm

($\leftarrow$) We have

$A_E \vdash \forall x(\theta(\overline{\#\rho}, \overline{\#\rho}, x) \leftrightarrow x = \overline{\#\rho(\overline{\#\rho})})$	By (6)
$A_E \vdash \theta(\overline{\#\rho}, \overline{\#\rho}, x) \leftrightarrow x = \overline{\#\sigma}$	by Right $\forall$-Elim
$A_E, \theta(\overline{\#\rho}, \overline{\#\rho}, x) \vdash x = \overline{\#\sigma}$	By Ded Thm
$A_E, \theta(\overline{\#\rho}, \overline{\#\rho}, x) \vdash x = \overline{\#\sigma} \to (\phi(\overline{\#\sigma}) \to \phi)$	By Ax(H8) and = Reflexivity
$A_E, \theta(\overline{\#\rho}, \overline{\#\rho}, x) \vdash \phi(\overline{\#\sigma}) \to \phi$	By Ax(H8) and = Reflexivity
$A_E, \phi(\overline{\#\sigma}) \vdash \theta(\overline{\#\rho}, \overline{\#\rho}, x) \to \phi$	By Ded Thm twice
$A_E, \phi(\overline{\#\sigma}) \vdash \forall x(\theta(\overline{\#\rho}, \overline{\#\rho}, x) \to \phi)$	By Gen
$A_E \vdash \phi(\overline{\#\sigma}) \to \sigma$	by Ded Thm

□

Theorem 33 (Tarski's Undefinability Theorem)**.** $\#Th(\mathfrak{N})$ is undefinable in $\mathfrak{N}$

Proof. Suppose for contradiction that $\#Th(\mathfrak{N})$ is defined by some $\alpha \in Fml_{\mathcal{L}_{\mathbb{N}}}$. By applying the fixed point lemma for $(\neg\alpha)$, we get some $\sigma \in Sent_{\mathcal{L}_{\mathbb{N}}}$. Therefore :

$$\mathfrak{N} \models \sigma \iff \#\sigma \in \#\mathrm{Th}(\mathfrak{N}) \iff \mathfrak{N} \models \alpha(\#\sigma) \iff \mathfrak{N} \models \neg\sigma \text{ ↯}$$

□

Definition 34. In a language $\mathcal{L}$, a theory T is ω-inconsistent iff there exists $\phi \in Fml_{\mathcal{L}}$ and $x \in Var$ s.t : $FV(\phi) = \{x\}$, for all $n \in \mathbb{N}$: $T \vdash \phi(\overline{n})$ and $T \vdash \exists x \neg\phi(x)$.
And T is ω-consistent when it's not ω-inconsistent.

Lemma 35. For any $\mathcal{L}$-theory T : T is ω-consistent $\Rightarrow$ T is consistent

Proof. This comes naturally from the contrapositive : Suppose T is inconsistent, then for all $\theta \in Fml_{\mathcal{L}}$, $T \vdash \theta$ so T clearly satisfy the definition of ω-inconsistency. □

Theorem 36 (Gödel First Incompleteness Theorem)**.** Let $\Gamma \subset Fml_{\mathcal{L}_{\mathbb{N}}}$ with $\#\Gamma$ recursive. Here are different versions :

1. $\mathfrak{N} \models \Gamma \Rightarrow \mathrm{Cn}\Gamma$ incomplete
2. Γ ω-consistent $\Rightarrow \mathrm{Cn}\Gamma$ incomplete
3. Γ consistent $\Rightarrow \mathrm{Cn}\Gamma$ incomplete

Proof.

1. Suppose for contradiction that $\mathrm{Cn}\Gamma$ is complete $\Rightarrow \#\mathrm{Cn}\Gamma$ is recursive by Theorem 29.(14) and $\mathrm{Cn}\Gamma = \mathrm{Th}\mathfrak{N}$ since we had $\mathrm{Cn}\Gamma \subseteq \mathrm{Th}\mathfrak{N}$. So $\#\mathrm{Th}\mathfrak{N}$ is recursive and by Corollary 31 $\#\mathrm{Th}\mathfrak{N}$ is definable in $\mathfrak{N}$ ↯ contradicting Tarski's Thm.

2. By Theorem 29.(15), as Γ is recursive we have that $Proof_{\Gamma}$ is recursive too and thus represented by some $\mathrm{Proof}_{\Gamma} \in Fml_{\mathcal{L}_{\mathbb{N}}}$ with $FV(\mathrm{Proof}_{\Gamma}) \subseteq \{x, y\} \subseteq Var$.
Let $\mathrm{Prv}_{\Gamma} := \exists x \mathrm{Proof}_{\Gamma}$, which is called the provability function, and which states that there exists a proof of its argument.
Let $G_{\Gamma} \in Sent_{\mathcal{L}_{\mathbb{N}}}$ be obtained from the fixed point lemma applied to $\neg\mathrm{Prv}_{\Gamma}$. Note that for $\theta \in Fml_{\mathcal{L}_{\mathbb{N}}}$:

$$\begin{aligned} \Gamma \vdash \theta &\Rightarrow \text{there exists a deduction } \mathcal{D} \text{ of } \theta \text{ from } \Gamma \\ &\Rightarrow A_E \vdash \mathrm{Proof}_{\Gamma}(\overline{\#\mathcal{D}}, \overline{\#\theta}) \\ &\Rightarrow A_E \vdash \mathrm{Prv}_{\Gamma}(\overline{\#\theta}) \text{ by Right } \exists\text{-Intro} \end{aligned} \tag{*}$$

   - If $\Gamma \vdash G_{\Gamma}$:

$$\Gamma\ \omega\text{-consistent} \overset{\text{Lemma 35}}{\Rightarrow} \Gamma \text{ consistent } \Rightarrow \Gamma \cup A_E \text{ consistent}$$

$$\begin{array}{ll} \Gamma \cup A_E \vdash G_{\Gamma} & \text{By Monicity} \\ \Gamma \cup A_E \vdash \mathrm{Prv}_{\Gamma}(\overline{\#G_{\Gamma}}) & \text{by (*)} \\ \Gamma \cup A_E \vdash \neg\mathrm{Prv}_{\Gamma}(\overline{\#G_{\Gamma}}) & \text{by the def of } G_{\Gamma} \end{array}$$

   Contradicting the consistency of $\Gamma \cup A_E \Rightarrow \Gamma \nvdash G_{\Gamma}$

- If $\Gamma \vdash \neg G_\Gamma$: This is equivalent as supposing that

$$\Gamma \vdash \mathrm{Prv}_\Gamma(\overline{\#G_\Gamma})$$
$$\text{or in other words } \Gamma \vdash \exists x \mathrm{Proof}_\Gamma(x, \overline{\#G_\Gamma})$$

$$\begin{aligned}\Gamma \text{ is } \omega\text{-consistent } &\Rightarrow \text{ it is not the case that for all } n \in \mathbb{N} : \Gamma \vdash \neg\mathrm{Proof}_\Gamma(\overline{n}, \overline{\#G_\Gamma}) \\ &\Rightarrow \text{there exists } m \in \mathbb{N} : \Gamma \nvdash \neg\mathrm{Proof}_\Gamma(\overline{n}, \overline{\#G_\Gamma})\end{aligned}$$

But at the same time we prove $\Gamma \nvdash G_\Gamma$

$$\begin{aligned}&\Rightarrow \langle n, \#G_\Gamma\rangle \notin Proof_\Gamma \\ &\Rightarrow \Gamma \vdash \neg\mathrm{Proof}_\Gamma(\overline{n}, \overline{\#G_\Gamma}) \ \lightning\end{aligned}$$

Contradicting the consistency of Γ (by Lemma 35 since Γ is ω-consistent)

3. This last proof is the most complex one and to perform it, we're going to use Rosser's trick which allows to require only the consistency by using another provability function.
As in the previous case we have that $Proof_\Gamma$ is represented by some $\mathrm{Proof}_\Gamma \in Fml_{\mathcal{L}_\mathbb{N}}$ with $FV(\mathrm{Proof}_\Gamma) \subseteq \{x, y\} \subseteq Var$.
We want to create a function Prv_Γ^R stating that if its argument has a proof, then its negation has a proof with a smaller code. So we will need a negation function

$$neg : \mathbb{N} \to \mathbb{N} : neg(y) := \ll \#(, \#\neg \gg *y* \ll \#) \gg$$

i.e if $y = \#\phi$, some $\phi \in Fml_{\mathcal{L}_\mathbb{N}}$, then $neg(y) = \#(\neg\phi)$.
By Theorem 25, neg is functionally represented by some $\mathrm{Neg} \in Fml_{\mathcal{L}_\mathbb{N}}$, and w.l.o.g $FV(\mathrm{Neg}) \subseteq \{x, y\} \subseteq Var$ (fresh variables).

$$\text{so for all } a \in \mathbb{N} : A_E \vdash \forall y(\mathrm{Neg}(\overline{a}, y) \leftrightarrow y = \overline{neg(a)})$$
$$\text{and for all } \phi \in Fml_{\mathcal{L}_\mathbb{N}} : A_E \vdash \forall y(\mathrm{Neg}(\overline{\#\phi}, y) \leftrightarrow y = \overline{\#(\neg\phi)})$$

Let

$$\mathrm{Prv}_\Gamma^R(y) := \exists x(\mathrm{Proof}_\Gamma(x, y) \wedge \exists a(\mathrm{Neg}(y, a) \wedge \neg\exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, a))))$$

and let $\mathcal{R}$, called the Rosser sentence, be the result of the Fixed-Point Lemma applied to $\neg\mathrm{Prv}_\Gamma^R$:

$$A_E \vdash \mathcal{R} \leftrightarrow \neg\mathrm{Prv}_\Gamma^R(\overline{\#\mathcal{R}})$$

- Suppose for contradiction that $\Gamma \vdash \mathcal{R}$

$$\begin{aligned}\Gamma \vdash \mathcal{R} &\Rightarrow \text{there exists a deduction } \mathcal{D} \text{ of } \mathcal{R} \text{ from } \Gamma \\ &\Rightarrow \langle \#\mathcal{D}, \#\mathcal{R}\rangle \in Proof_\Gamma \\ &\Rightarrow A_E \vdash \mathrm{Proof}_\Gamma(\overline{\#\mathcal{D}}, \overline{\#\mathcal{R}})\end{aligned}$$

So

$$
\begin{array}{ll}
A_E, \Gamma \vdash \text{Proof}_\Gamma(\overline{\#\mathcal{D}}, \overline{\#\mathcal{R}}) & \text{by Monicity} \\
A_E, \Gamma \vdash \mathcal{R} & \text{by Monicity} \\
A_E, \Gamma \vdash \neg\text{Prv}^R_\Gamma(\overline{\#\mathcal{R}}) & \text{by MP} \\
A_E, \Gamma \vdash \forall x(\text{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) \to \forall a(\text{Neg}(\overline{\#\mathcal{R}}, a) \to \exists b(b \leqslant x \wedge \text{Proof}_\Gamma(b, a)))) & \text{by rewritting} \\
A_E, \Gamma \vdash \text{Proof}_\Gamma(\overline{\#\mathcal{D}}, \overline{\#\mathcal{R}}) \to \forall a(\text{Neg}(\overline{\#\mathcal{R}}, a) \to \exists b(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, a))) & \text{by Right } \forall\text{-Elim} \\
A_E, \Gamma \vdash \forall a(\text{Neg}(\overline{\#\mathcal{R}}, a) \to \exists b(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, a))) & \text{by MP} \\
A_E, \Gamma \vdash \text{Neg}(\overline{\#\mathcal{R}}, \overline{\#(\neg\mathcal{R})}) \to \exists b(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) & \text{by Right } \forall\text{-Elim} \\
A_E, \Gamma \vdash \text{Neg}(\overline{\#\mathcal{R}}, \overline{\#(\neg\mathcal{R})}) & \text{by MP} \\
A_E, \Gamma \vdash \exists b(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) & \text{by MP} \\
A_E, \Gamma \vdash \neg\forall b \neg(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) & \text{by rewritting}
\end{array}
$$

Γ consistent and $\Gamma \vdash \mathcal{R} \Rightarrow \Gamma \nvdash \neg\mathcal{R}$. So for all $n \in \mathbb{N}$:

$$\langle n, \overline{\#(\neg\mathcal{R})} \rangle \notin Proof_\Gamma$$

$$
\begin{array}{rl}
A_E \vdash \neg\text{Proof}_\Gamma(\overline{n}, \overline{\#(\neg\mathcal{R})}) & \text{by def of Proof}_\Gamma \\
A_E \vdash \overline{n} = b \to (\neg\text{Proof}_\Gamma(\overline{n}, \overline{\#(\neg\mathcal{R})}) \to \neg\text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) & \text{by Ax(H8)} \\
A_E \vdash \overline{n} = b \to \neg\text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})}) & \text{by } \to\text{-Exchange and MP} \\
A_E, b = \overline{n} \vdash \neg\text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})}) & \text{by } =\text{-Sym and Ded Thm}
\end{array}
$$

Therefore

$$
\begin{array}{rl}
A_E, b = \overline{0} \vee \ldots \vee b = \overline{\#\mathcal{D}} \vdash \neg\text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})}) & \text{by Left } \vee\text{-Intro} \\
A_E \vdash b = \overline{0} \vee \ldots \vee b = \overline{\#\mathcal{D}} \to \neg\text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})}) & \text{by Ded Thm} \\
A_E \vdash b \leqslant \overline{\#\mathcal{D}} \to b = \overline{0} \vee \ldots \vee b = \overline{\#\mathcal{D}} & \text{by Lemma 8} \\
A_E \vdash b \leqslant \overline{\#\mathcal{D}} \to \neg\text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})}) & \text{by } \to\text{-Transitivity} \\
A_E \vdash \neg(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) & \text{by def of } \wedge \\
A_E, \Gamma \vdash \forall b \neg(b \leqslant \overline{\#\mathcal{D}} \wedge \text{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) \ \lightning & \text{by Gen and Moni}
\end{array}
$$

Contradicting the consistency of $\Gamma \cup A_E$.
$\Rightarrow \Gamma \nvdash \mathcal{R}$

- Suppose for contradiction that $\Gamma \vdash \neg\mathcal{R}$

$$
\begin{aligned}
\Gamma \vdash \mathcal{R} &\Rightarrow \text{there exists a deduction } \mathcal{D} \text{ of } (\neg\mathcal{R}) \text{ from } \Gamma \\
&\Rightarrow \langle \#\mathcal{D}, \#(\neg\mathcal{R}) \rangle \in Proof_\Gamma \\
&\Rightarrow A_E \vdash \text{Proof}_\Gamma(\overline{\#\mathcal{D}}, \overline{\#(\neg\mathcal{R})})
\end{aligned}
$$

And

$$\begin{aligned}
A_E, \overline{\#\mathcal{D}} \leqslant x &\vdash \overline{\#\mathcal{D}} \leqslant x && \text{by Ax}\\
A_E, \overline{\#\mathcal{D}} \leqslant x &\vdash \mathrm{Proof}_\Gamma(\overline{\#\mathcal{D}}, \overline{\#(\neg\mathcal{R})}) && \text{by Monicity}\\
A_E, \overline{\#\mathcal{D}} \leqslant x &\vdash \overline{\#\mathcal{D}} \leqslant x \wedge \mathrm{Proof}_\Gamma(\overline{\#\mathcal{D}}, \overline{\#(\neg\mathcal{R})}) && \text{by Right } \wedge\text{-Intro}\\
A_E, \overline{\#\mathcal{D}} \leqslant x &\vdash \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) && \text{by Right } \exists\text{-Intro}\\
A_E &\vdash \overline{\#\mathcal{D}} \leqslant x \rightarrow \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) && \text{by Ded Thm} \qquad (*)
\end{aligned}$$

$\Gamma \vdash \neg\mathcal{R}$ and Γ consistent $\Rightarrow \Gamma \nvdash \mathcal{R}$. So for all $n \in \mathbb{N}$:

$$\langle n, \#\mathcal{R}\rangle \notin \mathit{Proof}_\Gamma$$

$$\begin{aligned}
A_E &\vdash \neg\mathrm{Proof}_\Gamma(\overline{n}, \overline{\#\mathcal{R}}) && \text{by def of } \mathrm{Proof}_\Gamma\\
A_E &\vdash \overline{n} = x \rightarrow (\neg\mathrm{Proof}_\Gamma(\overline{n}, \overline{\#\mathcal{R}}) \rightarrow \neg\mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}})) && \text{by Ax(H8)}\\
A_E &\vdash \overline{n} = x \rightarrow \neg\mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) && \text{by } \rightarrow\text{-Exchange and MP}\\
A_E, x = \overline{n} &\vdash \neg\mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) && \text{by } =\text{-Sym and Ded Thm}
\end{aligned}$$

Therefore

$$\begin{aligned}
A_E &\vdash x = \overline{0} \vee \ldots \vee x = \overline{\#\mathcal{D}} \rightarrow \neg\mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) && \text{by Left } \vee\text{-Intro and Ded Thm}\\
A_E &\vdash x \leqslant \overline{\#\mathcal{D}} \rightarrow x = \overline{0} \vee \ldots \vee x = \overline{\#\mathcal{D}} && \text{by Lemma 8}\\
A_E &\vdash x \leqslant \overline{\#\mathcal{D}} \rightarrow \neg\mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) && \text{by } \rightarrow\text{-Transitivity}\\
A_E &\vdash \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) \rightarrow \neg(x \leqslant \overline{\#\mathcal{D}}) && \text{by } \rightarrow\text{-Transitivity}\\
A_E &\vdash x \leqslant \overline{\#\mathcal{D}} \vee \overline{\#\mathcal{D}} \leqslant x && \text{by (L3) and Right } \forall\text{-Elim twice}\\
A_E &\vdash \neg(x \leqslant \overline{\#\mathcal{D}}) \rightarrow \overline{\#\mathcal{D}} \leqslant x && \text{by def of } \vee\\
A_E &\vdash \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) \rightarrow \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) && \text{by (*) and } \rightarrow\text{-Trans} \quad (**)
\end{aligned}$$

And finally

$$\begin{aligned}
A_E, \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}), \mathrm{Neg}(\overline{\#\mathcal{R}}, a) &\vdash \mathrm{Neg}(\overline{\#\mathcal{R}}, a) && \text{by Ax}\\
\text{——}\shortparallel\text{——} &\vdash a = \overline{\#(\neg\mathcal{R})} && \text{since Neg defines neg}\\
\text{——}\shortparallel\text{——} &\vdash \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) && \text{by Ax}\\
\text{——}\shortparallel\text{——} &\vdash \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, \overline{\#(\neg\mathcal{R})})) && \text{by (**) and MP}\\
\text{——}\shortparallel\text{——} &\vdash \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, a)) && \text{by Ax(H8) and MP twice}
\end{aligned}$$

Thus

$$\begin{aligned}
A_E, \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) &\vdash \mathrm{Neg}(\overline{\#\mathcal{R}}, a) \rightarrow \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, a)) && \text{by Ded Thm}\\
A_E, \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) &\vdash \forall a(\mathrm{Neg}(\overline{\#\mathcal{R}}, a) \rightarrow \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, a))) && \text{by Gen}
\end{aligned}$$

And

$$\begin{aligned}
A_E &\vdash \mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) \rightarrow \forall a(\mathrm{Neg}(\overline{\#\mathcal{R}}, a) \rightarrow \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, a))) && \text{by Ded Thm}\\
A_E &\vdash \forall x(\mathrm{Proof}_\Gamma(x, \overline{\#\mathcal{R}}) \rightarrow \forall a(\mathrm{Neg}(\overline{\#\mathcal{R}}, a) \rightarrow \exists b(b \leqslant x \wedge \mathrm{Proof}_\Gamma(b, a)))) && \text{by Gen}\\
A_E &\vdash \neg\mathrm{Prv}^R_\Gamma(\overline{\#\mathcal{R}}) && \text{by rewritting}\\
A_E &\vdash \mathcal{R} && \text{by MP}
\end{aligned}$$

Therefore $A_E, \Gamma \vdash \mathcal{R}$ and $A_E, \Gamma \vdash \neg\mathcal{R}$ $\lightning$ contradicting the consistency.
$\Rightarrow \Gamma \nvdash \neg\mathcal{R}$.

This means that $\mathcal{R}$ is an undecidable sentence for Γ, which is incomplete.

$\square$

6 Conclusion & Bibliography

When I asked Dr Fujimoto to do a project under his supervision and he mentioned the possibility to study Gödel's work further into details, my curiosity was immediately aroused. Thanks to his amazing Logic lectures, I already had beliefs of how great it could be.

The fact that the books used as references had different notations and did not go much into detail, often giving the idea of a resolution only and sometimes even omitting proofs to let them as exercises for the reader, can seem like an annoying constraint. However it was in fact greatly appreciated since, most of the time, I would then just take an empty sheet of paper, copy an assertion to prove and then find a proof of my one.
Even though I have to admit I had to occasionally give a few glances in the sources, I managed to do it as rarely as possible. As an example, for the Rosser's trick I only looked at how to construct the Rosser sentence and checked how to progress further once, since I was stuck where one has to make use of Lemma 8.

The way of proceeding was really entertaining, and in general the whole project helped me discovering even more how deep and interesting Logic can be.
I am satisfied of how much work has been carried out, and my only regret is that more could have been done in a context without time constraint.

I can without a doubt express my satisfaction about the experience and would like to thank Dr Kentaro Fujimoto for accepting my request, for giving his time for me when it was needed and for being a wonderful supervisor.

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