ÉCOLE POLYTECHNIQUE
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Master semester project

Further topics in Logic & Vaught Theorem

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Contents

Chapter 1

Introduction

This document is a 1st year graduate/Master semester project, written by Aloïs Rosset at the École Polytechnique Fédérale de Lausanne (EPFL), under the guidance of Dr Jacques Duparc and under the supervision of the PhDs Gianluca Basso and Louis Vuilleumier. It is aimed at any student who is interested in what can be explored beyond the usual first contact with Logic.

In the first place, the three first classic results of logic, namely completeness, Löwenheim-Skolem and compactness theorems will be revisited using each time different tools from different fields. Completeness will be demonstrated through topology. Löwenheim-Skolem will be deduced from the point of view of game theory. Compactness will be proved with the use of ultrafilters and ultraproducts.

In the second place, we are going to dive deeper into model theory in order to reach the famous Vaught theorem. This will require a study of different notions, namely homogeneous, saturated and special structures. Many properties of those concepts will be given to get a better grasp of their meaning, while building all the knowledge needed to demonstrate the result we are aiming for.

Here are the notation conventions that are going to be used.

Notation. Let $\mathcal{L}$ be a first-order language and $\mathcal{M}$ be an $\mathcal{L}$-structure. We will denote by the corresponding capital Roman letter M the universe of $\mathcal{M}$. The sets Var, $\mathrm{Tm}_{\mathcal{L}}$, $\mathrm{Fml}_{\mathcal{L}}$, $\mathrm{Sent}_{\mathcal{L}}$ and $\mathrm{Th}_{\mathcal{L}}(\mathcal{M})$ will denote respectively the set $\{v_0, v_1, \ldots\}$ of variables in $\mathcal{L}$, the set of $\mathcal{L}$-terms, the set of $\mathcal{L}$-formulae, the set of $\mathcal{L}$-sentences and the set of $\mathcal{L}$-sentences true in $\mathcal{M}$. By an $\mathcal{L}$-theory, we will mean a set of $\mathcal{L}$-sentences, or in other words a subset of $\mathrm{Sent}_{\mathcal{L}}$. With eventually indices, variables will be denoted by v, x, y, z, terms by t, constant symbols by c, d, function symbols by f, g, h, predicate symbols by P, R, formulae by ϕ, ψ, θ, χ, and sentences by σ, ρ, τ.

Let $t \in \mathrm{Tm}_{\mathcal{L}}$ and $\phi \in \mathrm{Fml}_{\mathcal{L}}$. Then $\mathrm{FV}(\phi)$ will denote the set of free variables of ϕ. Writing $t(x_1, \ldots, x_n)$ will express when $x_1, \ldots, x_n \in \mathrm{Var}$ and all the variables occurring in t are among those. Writing $\phi(x_1, \ldots, x_n)$ will express when $x_1, \ldots, x_n \in \mathrm{Var}$ and $\mathrm{FV}(\phi) \subseteq \{x_1, \ldots, x_n\}$. Moreover, when we have elements $a_1, \ldots, a_n \in M$, $\mathcal{M} \vDash \phi(a_1, \ldots, a_n)$ will denote the truth of ϕ

in $\mathcal{M}$ under the interpretation $x_i \mapsto a_i$ for $1 \leqslant i \leqslant n$. More generally, a function $s : \mathrm{Var} \to M$ will be called a *variable assignment.* For some $x \in \mathrm{Var}$ and $a \in M$, we will use the notation

$$s\frac{a}{x} : \mathrm{Var} \to M : y \mapsto \begin{cases} a \text{ if } y = x \\ s(y) \text{ otherwise.} \end{cases}$$

Furthermore, $\mathcal{M} \vDash_s \phi$ will be used to state when $\mathcal{M} \vDash \phi(s(x_1), \ldots, s(x_n))$.

Concerning set theory notations, let X, Y be sets. The sets $\mathcal{P}(X)$, $\mathcal{P}_{<\omega}(X)$, $\mathcal{P}_{\omega}(X)$ and ${}^{X}Y$ will denote respectively the power set of X, the set of finite subsets of X, the set of countable subsets of X and the set of functions $X \to Y$. The *ordinals* will be symbolized using letters at the beginning of the Greek alphabet: $\alpha, \beta, \gamma, \ldots$. Whereas for *cardinals* it is middle of the alphabet letters that will be employed: $\kappa, \lambda, \mu, \ldots$. The *equipollence* relation to express when a bijection exists will be denoted $X \approx Y$, and for injections it will be $X \preccurlyeq Y$.

The framework of this project will assume and use the axiom of choice and some of its equivalent forms, namely Zorn's lemma and the well-ordering principle.

Chapter 2

Classical results of logic with different approaches

2.1 Topology and completeness theorem

The aim of this section is to present one way of benefiting from topology in logic. For that, we will provide a proof of the completeness theorem by endowing some theories with specific topologies and then making use of Baire category theorem and Urysohn's metrization theorem. To this end, let us first recall those two classical results.

Theorem 2.1 (Baire Category Theorem). *Every complete metric space (M, d) is a* Baire space*, meaning that for any family $\mathcal{U}$ of open dense subsets of M, their intersection $\bigcap \mathcal{U}$ is dense in M.*

Theorem 2.2 (Urysohn's Metrization Theorem). *Every Hausdorff second-countable regular space is metrizable.*

Theorem 2.3 (Completeness Theorem). *Let $\mathcal{L}$ be a countable first-order language and let $\tilde{T}$ be a theory over $\mathcal{L}$. If $\tilde{T}$ is consistent, then $\tilde{T}$ is satisfiable.*

Proof. Let us first treat the case where $\mathcal{L}$ is a relational language without equality. Let

$$\mathcal{L}' = \mathcal{L} \cup \{c_0, c_1, \ldots\},$$
$$X = \{\text{theories } T \subseteq \text{Sent}_{\mathcal{L}'} \mid T \text{ is maximal}\},$$

where $c_0, c_1, \ldots$ are new constants symbols. Recall that a theory T is maximal if for each $\mathcal{L}'$-sentence σ, exactly one of σ and $\neg\sigma$ belongs to T. For each finite theory $\Delta \subseteq \text{Sent}_{\mathcal{L}'}$, define the basic open subset $U_\Delta = \{T \in X \mid \Delta \subseteq T\}$. Let $\mathcal{T}$ be the topology with basis $\mathcal{B} = \{U_\Delta \subseteq X \mid \Delta \subseteq \text{Sent}_{\mathcal{L}'} \text{ finite}\}$.

Claim: $(X, \mathcal{T})$ is a compact metrizable topological space.

First, let us verify the two axioms for a topological basis:

1. Take $T \in X$. By maximality $T \neq \varnothing$, hence there exists an $\mathcal{L}'$-sentence $\sigma \in T$. Therefore $\{\sigma\} \subseteq T$ and $T \in U_{\{\sigma\}}$.
2. Let $U_{\Delta_1}, U_{\Delta_2} \in \mathcal{B}$. The condition that we want immediately follows from $U_{\Delta_1} \cap U_{\Delta_2} = U_{\Delta_1 \cup \Delta_2} \in \mathcal{B}$.

Hausdorffness: Take $T, T' \in X$ distinct. By maximality, there exists a $\sigma \in \mathrm{Sent}_{\mathcal{L}'}$ such that $\sigma \in T$ and $\neg\sigma \in T'$. Thus $T \in U_{\{\sigma\}}$ and $T' \in U_{\{\neg\sigma\}}$, which are disjoint basic open subsets of X.

Second countable: Since $\mathcal{L}'$ is countable, then so is $(\mathcal{L}')^{<\omega}$. Therefore, there are only countably many theories $\Delta \subseteq \mathrm{Sent}_{\mathcal{L}'} \subseteq (\mathcal{L}')^{<\omega}$ and countably many basic open subsets.

Compactness: Suppose we have an open cover of X with basic open subsets: $X = \bigcup_{n<\omega} U_{\Delta_n}$. Observe that for a fixed $\sigma \in \mathrm{Sent}_{\mathcal{L}'}$, the maximality forces any theory $T \in X$ to have either $\sigma \in T$ or $\neg\sigma \in T$. Therefore, take $\sigma \in \mathrm{Sent}_{\mathcal{L}}$ and $n < m < \omega$ so that $\sigma \in \Delta_n$ and $\neg\sigma \in \Delta_m$. Then $X = U_{\Delta_n} \cup U_{\Delta_m}$ gives a finite subcover.

As Hausdorff and compact imply regular, Urysohn's metrization theorem give us the claim. Now, let

$$Y = \left\{ T \in X \;\middle|\; \tilde{T} \subseteq T \text{ and } T \nvdash \perp \text{ i.e. consistent} \right\}.$$

Observe that Y is closed in X, since

$$\begin{aligned} X \setminus Y &= \left\{ T \in X \;\middle|\; \tilde{T} \nsubseteq T \text{ or } T \vdash \perp \right\} \\ &= \bigcup \left\{ U_{\{\sigma\}} \mid \sigma \in \mathrm{Sent}_{\mathcal{L}'} : \sigma \notin T \right\} \cup \bigcup \left\{ U_{\{\sigma,\rho\}} \mid \sigma, \rho \in \mathrm{Sent}_{\mathcal{L}'} : \sigma \vdash \neg\rho \right\}. \end{aligned}$$

We have $Y \subseteq X$ closed and X compact and metrizable. Therefore Y is also compact and metrizable and thus complete. Observe that Y is non-empty, as we can start with $T = \tilde{T}$ and repeatedly add $\sigma \in \mathrm{Sent}_{\mathcal{L}'}$ every time that $T \nvdash \neg\sigma$, i.e., that $T \cup \{\sigma\}$ is consistent.

Given a formula $\phi(x) \in \mathrm{Fml}_{\mathcal{L}'}$, define the property (P_ϕ) by:

$$T \subseteq \mathrm{Sent}_{\mathcal{L}'} \text{ satisfies } (P_\phi) :\Longleftrightarrow \exists x\phi \in T \text{ implies } \phi[\tfrac{c}{x}] \in T \text{ for some constant } c \in \mathcal{L}' \setminus \mathcal{L}.$$

For the property (P_ϕ), let $G_\phi = \{T \in X \mid T \text{ satisfies } (P_\phi)\}$. Observe that G_ϕ is open in X as

$$G_\phi = \bigcup_{n<\omega} \left(U_{\{\exists x\phi, \phi[\frac{c_n}{x}]\}} \cup U_{\{\neg\exists x\phi, \phi[\frac{c_n}{x}]\}} \cup U_{\{\neg\exists x\phi, \neg\phi[\frac{c_n}{x}]\}} \right),$$

which comes from the truth tables of the formulae $\exists x\phi \to \phi[\frac{c_n}{x}]$, for $n < \omega$.

Claim: $G_\phi \cap Y$ is dense in Y.

Take $\Delta \subseteq \mathrm{Sent}_{\mathcal{L}'}$ finite and $T \in U_\Delta \cap Y$. We want to prove the existence of some $S \in U_\Delta \cap Y \cap G_\phi$. If $\exists x\phi \notin T$, then T satisfies trivially (P_ϕ) making of $S = T$ a valid choice. Otherwise, suppose $\exists x\phi \in T$ and observe that $T' = \tilde{T} \cup \Delta$ already satisfies two of the conditions S needs to: $\Delta \subseteq T'$ and $\tilde{T} \subseteq T'$, so that we still need to satisfy (P_ϕ), consistency and maximality. Take $c \in \{c_0, c_1, \ldots\}$

a constant symbol not occurring in Δ nor in ϕ. We claim that $T' \cup \{\phi[\frac{c}{x}]\}$ is consistent. Indeed, suppose ad absurdum that $T' \cup \{\phi[\frac{c}{x}]\}$ is inconsistent. Hence $T', \phi[\frac{c}{x}] \vdash \bot$. Observe that for y a fresh variable not yet used, we have $T', \phi[\frac{y}{x}] \vdash \bot$. E.g. for Hilbert Calculus, if $\mathcal{D} = \langle \theta_1, \ldots, \theta_k \rangle$ is a deduction of $\theta_k = \bot$ from $T' \cup \{\phi[\frac{c}{x}]\}$, then $\mathcal{D}'$, obtained by replacing all occurrences of c by y in $\mathcal{D}$, is clearly a deduction of $\bot$ from $T' \cup \{\phi[\frac{y}{x}]\}$. Or in sequent calculus, a quick induction on the height of the demonstration tree of $T', \phi[\frac{c}{x}] \vdash \bot$, shows that by again replacing all occurrences of c by y, we get the demonstration tree of $T', \phi[\frac{y}{x}] \vdash \bot$. Therefore, by the left $\exists$ introduction rule, we have $T', \exists x\phi \vdash \bot$ and thus $T \vdash \bot$ since $T' \cup \{\exists x\phi\} = \tilde{T} \cup \Delta \cup \{\exists x\phi\} \subseteq T$, contradicting the consistency of T $\lightning$. Let S be the maximal consistent extension of $T' \cup \{\phi[\frac{c}{x}]\}$. This S is in $U_\Delta \cap Y \cap G_\phi$, as desired.

Let $\mathcal{U} = \{G_\phi \cap Y \mid \phi \in \mathrm{Fml}_{\mathcal{L}'}, FV(\phi) = \{x\}\}$. By Baire category theorem, there exists $T \in \bigcap \mathcal{U}$, meaning that $\tilde{T} \subseteq T$, $T \nvdash \bot$, T is maximal and T satisfies all properties (P_ϕ).

Let us exhibit a explicit model of T. Let $\mathcal{M}$ be the structure with $M = \mathbb{N}$, $c_i^{\mathcal{M}} = i$ for $i \in \mathbb{N}$. For an k-ary predicate symbol $R \in \mathcal{L}$, let $R^{\mathcal{M}} = \{(n_1, \ldots, n_k) \mid R(c_{n_1}, \ldots, c_{n_k}) \in T\}$, i.e., be defined by the atomic sentences of T. Let us verify that for $\sigma \in \mathrm{Sent}_{\mathcal{L}'}$: $\mathcal{M} \vDash \sigma \iff \sigma \in T$.

- Already defined to be true for atomic sentences.
- If σ is $\neg\rho$:

$$\begin{aligned}\mathcal{M} \vDash \sigma &\iff \mathcal{M} \nvDash \rho \\ &\overset{\text{I.H.}}{\iff} \rho \notin T \\ &\overset{\text{Maximality}}{\iff} \sigma \in T.\end{aligned}$$

- If σ is $\rho \wedge \tau$:

$$\begin{aligned}\mathcal{M} \vDash \sigma &\iff \mathcal{M} \vDash \rho \text{ and } \mathcal{M} \vDash \tau \\ &\overset{\text{I.H.}}{\iff} \rho \in T \text{ and } \tau \in T \\ &\underset{\text{Consistency}}{\overset{\text{Maximality}}{\iff}} \sigma \in T.\end{aligned}$$

- If σ is $\exists x\phi$:

$$\begin{aligned}\mathcal{M} \vDash \sigma &\iff \text{for some } n \in \mathbb{N} : \mathcal{M} \vDash \phi(n) \\ &\iff \text{for some } n \in \mathbb{N} : \mathcal{M} \vDash \phi[\tfrac{c_n}{x}] \\ &\overset{\text{I.H.}}{\iff} \text{for some } n \in \mathbb{N} : \phi[\tfrac{c_n}{x}] \in T \\ &\underset{\text{Property } (P_\phi)}{\overset{\text{Consistency}}{\iff}} \sigma \in T.\end{aligned}$$

This proved our theorem in the case of $\mathcal{L}$ being relational and without equality. Observe that the case with equality is a sub-case of when a binary predicate symbol is added to $\mathcal{L}$ and three sentences are added in $\tilde{T}$ requesting the relation to be reflexive, symmetric and transitive. And for every k-ary function symbols $f \in \mathcal{L}$ (where we consider constant symbols to be 0-ary function symbols), replace it by a predicate symbol $G_f \in \mathcal{L}$ which will represent its graph, and add the axiom $\forall x_1 \ldots \forall x_k \exists! y G_f(x_1, \ldots, x_k, y)$ to T. Then we can ignore the function symbol as we did in the proof and at the end, give the interpretation $f^{\mathcal{M}}(n_1, \ldots, n_k) \coloneqq$ the unique $m \in \mathbb{N}$ such that $(n_1, \ldots, n_k, m) \in G_f^{\mathcal{M}}$. □

2.2 Game Theory and Löwenheim-Skolem theorem

Game theory is a field which has strong links with logic. In this section we will exhibit one particular example, which is how the use of a specific game called the *cub game* can be of use to demonstrate the Löwenheim-Skolem theorem.

Definition 2.4. A *two-person game of perfection information* consists of two players, **I** and **II** (that we will call *he* and respectively *she*), picking elements of a set A alternatively, starting with player **I**, for a finite number n of rounds. Player **I** chooses some $a_0 \in A$, then **II** chooses some $b_0 \in A$. When a_i and b_i have been played for $0 \leqslant i < n-1$, **I** picks some $a_{i+1} \in A$ and **II** picks some $b_{i+1} \in A$. The game stops after the choice of a_{n-1} and b_{n-1}. The winner is decided by a set W consisting of all the winning situations for player **II**.

Mathematically, this means that for a set A, a natural number n and a subset $W \subseteq A^{2n}$, the game $\mathcal{G}_n(A, W)$ is as such: any two sequences $\overline{a} = (a_0, \ldots, a_{n-1})$ and $\overline{b} = (b_0, \ldots, b_{n-1})$ of elements of A, which are called the *play* of **I**, respectively of **II**, form a *play* of $\mathcal{G}_n(A, W)$:

$$(\overline{a}; \overline{b}) \coloneqq (a_0, b_0, \ldots, a_{n-1}, b_{n-1}).$$

A play is a *win* for **II** if $(\overline{a}, \overline{b}) \in W$, otherwise it is a *win* for **I**.

I	**II**
a_0	
	b_0
a_1	
	b_1
$\vdots$	
	$\vdots$
a_n	
	b_n

Fig. 2.1: Finite Game $\mathcal{G}_n(A, W)$

Definition 2.5. Given a game $\mathcal{G}_n(A, W)$, a *strategy* of player **I** is a sequence

$$\sigma = (\sigma_0, \ldots, \sigma_{n-1})$$

of functions $\sigma_i : A^i \to A$, $(0 \leqslant i \leqslant n-1)$. Player **I** has *used strategy σ in the play* $(\overline{a}; \overline{b})$ if

$$a_0 = \sigma_0, \text{ which is purely a number,}$$
$$a_i = \sigma_i(b_0, \ldots, b_{i-1}) \text{ for } 1 \leqslant i \leqslant n-1.$$

A *strategy* of player **II** is a sequence

$$\tau = (\tau_0, \ldots, \tau_{n-1})$$

of functions $\tau_i : A^{i+1} \to A$, $(0 \leqslant i \leqslant n-1)$. Player **II** has *used strategy τ in the play* $(\overline{a}; \overline{b})$ if

$$b_i = \tau_i(a_0, \ldots, a_i) \text{ for } 0 \leqslant i \leqslant n-1.$$

A strategy for one of the two player is a *winning strategy* if every play where he, respectively she, has used the strategy ends up with a win for him, respectively her.

Definition 2.6. Let us extend the notion of game to allow for countably infinite ones. Let A be a set, n be a natural number and $W \subseteq A^{\mathbb{N}}$. The game $\mathcal{G}_\omega(A, W)$ is as such: any two sequences $\overline{a} = (a_0, \ldots, a_n)$ and $\overline{b} = (b_0, \ldots, b_n)$ of elements of A, which are called the *play* of **I**, respectively of **II**, form a *play* of $\mathcal{G}_n(A, W)$:

$$(\overline{a}; \overline{b}) = (a_0, b_0, a_1, b_1, \ldots).$$

A play is a *win* for **II** if $(\overline{a}, \overline{b}) \in W$, otherwise it is a *win* for **I**.

I	**II**
a_0	
	b_0
a_1	
	b_1
$\vdots$	$\vdots$

Fig. 2.2: Infinite Game $\mathcal{G}_\omega(A, W)$

Definition 2.7. Given an infinite game $\mathcal{G}_\omega(A, W)$, what we call a *strategy* of player **I** is, similarly as in the finite case, an infinite sequence

$$\sigma = (\sigma_0, \sigma_1, \ldots)$$

of functions $\sigma_i : A^i \to A$, where $i \in \mathbb{N}$. Player **I** has *used strategy σ in the play* $(\overline{a}; \overline{b})$ if

$$a_0 = \sigma_0, \text{ which is purely a number,}$$
$$a_i = \sigma_i(b_0, \ldots, b_{i-1}) \text{ for } i \in \mathbb{N}_{\geqslant 1}.$$

A *strategy* of player **II** is an infinite sequence

$$\tau = (\tau_0, \tau_1, \ldots)$$

of functions $\tau_i : A^{i+1} \to A$, where $i \in \mathbb{N}$. Player **II** has *used strategy* τ *in the play* $(\bar{a}; \bar{b})$ if

$$b_i = \tau_i(a_0, \ldots, a_i) \text{ for } i \in \mathbb{N}.$$

A strategy for one of the two player is a *winning strategy* if every play where he, respectively she, has used the strategy ends up with a win for him, respectively her.

The game used later to demonstrate Löwenheim-Skolem is defined as follows.

Definition 2.8. Let A and $\mathcal{C} \subseteq \mathcal{P}_\omega(A)$ be sets. We define the *Cub Game of* $\mathcal{C}$ as $G_{cub}(\mathcal{C}) :=$ $\mathcal{G}_\omega(A, W)$, where $W = \{(x_0, x_1, x_2, \ldots) \mid \{x_n \mid n \in \mathbb{N}\} \in \mathcal{C}\}$.

This means that in $G_{cub}(\mathcal{C})$, player **II** is consistently trying to keep the set of chosen elements to be in $\mathcal{C}$, while **I** tries to do the opposite. E.g. if $\mathcal{C} = \varnothing$, player **I** will always win, and conversely player **II** will necessarily be the winner when $\mathcal{C} = \mathcal{P}_\omega(A)$.

Lemma 2.9. *Let* $\mathcal{F} = \{f \mid f : A^{n_f} \to A\}$ *be a countable set of functions and*

$$\mathcal{C} := \{X \in \mathcal{P}_\omega(A) \mid X \text{ is closed under each } f \in \mathcal{F}\}.$$

*Then player **II** has a winning strategy in* $G_{cub}(\mathcal{C})$*.*

Proof. The idea of the strategy is quite clear, player **II** has to picks its element such that it periodically go through to all possible $f \in \mathcal{F}$ and all the elements that have already been picked, so that every possible combination is eventually played. We will use the notation as in the following figure.

I	**II**
a_0	
	a_1
a_2	
	a_3
$\vdots$	$\vdots$

Let us enumerate our functions: $\mathcal{F} = \{f_i \mid i \in \mathbb{N}\}$. For an natural number $m \in \mathbb{N}$, observe that if its prime decomposition turns out to be of the form

$$m = p_0^{m_0+1} \ldots p_k^{m_k+1},$$

where $p_0 = 2, p_1 = 3, \ldots$ and where k is the arity of f_{m_0}, then it can be used to describe the strategy for player **II** in order to choose its $(m+1)^{\text{th}}$ element:

$$a_{2m+1} = f_{m_0}(a_{m_1}, \ldots, a_{m_k}).$$

And if we are not in such a particular case, just let a_{2m+1} be a fixed element, say some $b \in A$. We observe that by our definition of the strategy, every possible image of some picked elements by some $f \in \mathcal{F}$ corresponds to a number $m \in \mathbb{N}$ and this is picked as a_{2m+1}. Therefore, any play $(a_0, a_1, \ldots)$ of this game with **II** using this strategy ends up with $\{a_0, a_1, \ldots\} \in \mathcal{C}$ and with a win for her. □

Lemma 2.10. *Let $\mathcal{L}$ be a countable language, $\mathcal{M}$ be an $\mathcal{L}$-structure and $\mathcal{C}$ be the set of universes of countable submodels of $\mathcal{M}$. Then player **II** has a winning strategy in $G_{cub}(\mathcal{C})$.*

Proof. It follows from the previous lemma and the followings simple observation:

$$\begin{aligned} \mathcal{C} &= \{N \mid \mathcal{N} \subseteq \mathcal{M} \text{ countable submodel}\} \\ &= \left\{X \in \mathcal{P}_\omega(M) \;\middle|\; X \text{ is closed under } \mathcal{F} := \left\{f^{\mathcal{M}} \mid f \text{ function symbol of } \mathcal{L}\right\}\right\}. \end{aligned}$$

□

Lemma 2.11. *Let A be a set. For each $n \in \mathbb{N}$, take $\mathcal{C}_n \subseteq \mathcal{P}_\omega(A)$. If player **II** has a winning strategy in $G_{cub}(\mathcal{C}_n)$, for each $n \in \mathbb{N}$, then she has one in $G_{cub}(\bigcap_{n\in\mathbb{N}} \mathcal{C}_n)$.*

Proof. For $n \in \mathbb{N}$, let us denote plays of $G_{cub}(\mathcal{C}_n)$ by $(a_0^n, b_0^n, a_1^n, b_1^n, \ldots)$ and plays of $G_{cub}(\bigcap_{n\in\mathbb{N}} \mathcal{C}_n)$ by $(a_0, b_0, a_1, b_1, \ldots)$.

I	II
a_0^n	
	b_0^n
a_1^n	
	b_1^n
$\vdots$	$\vdots$

Fig. 2.3: Game $G_{cub}(\mathcal{C}_n)$

I	II
a_0	
	b_0
a_1	
	b_1
$\vdots$	$\vdots$

Fig. 2.4: Game $G_{cub}(\bigcap_{n\in\mathbb{N}} \mathcal{C}_n)$

The idea of the strategy for player **II** is to eventually use all the strategies at disposal with every finite sequence $(a_0, b_0, \ldots, a_k)$ and $(a_0, b_0, \ldots, a_k, b_k)$ possible, where $k \in \mathbb{N}$. For that she can simply simulate all the games $G_{cub}(\mathcal{C}_n)$ for $n \in \mathbb{N}$ progressively with

$$a_{2j}^n = a_j \text{ and } a_{2j+1}^n = b_j \text{ for } j \in \mathbb{N},$$

as the main game advances. And her answers just have to browse all the simulated answers obtained by the existing strategies. For that we will need a bijection between $\mathbb{N} \times \mathbb{N}$ and $\mathbb{N}$, e.g.,

$$\pi : \mathbb{N} \times \mathbb{N} \to \mathbb{N} : (x, y) \mapsto \frac{(x+y)(x+y+1)}{2} + y.$$

With this bijection, we now have an explicit formulation for a winning strategy:

$$b_0 = b_{\pi(0,0)} = b_0^0,\ b_1 = b_{\pi(0,1)} = b_1^0, \ldots \text{ and in general } b_{\pi(n,k)} = b_k^n.$$

□

Lemma 2.12. *Let A be a set. For each $a \in A$, take $\mathcal{C}_a \subseteq \mathcal{P}_\omega(A)$. Recall the definition of the diagonal intersection:*

$$\mathop{\Delta}_{a\in A} \mathcal{C}_a := \left\{ X \in \mathcal{P}_\omega(A) \;\middle|\; X \in \bigcap_{a\in X} \mathcal{C}_a \right\}.$$

*If player **II** has a winning strategy in $G_{cub}(\mathcal{C}_a)$, for each $a \in A$, then she has one in $G_{cub}(\Delta_{a\in A}\,\mathcal{C}_a)$.*

Proof. Let us use the notation of Figure 2.5 for $G_{cub}(\mathcal{C}_{a_i})$, of Figure 2.6 for $G_{cub}(\mathcal{C}_{b_i})$ and of Figure 2.7 for $G_{cub}(\Delta_{a\in A}\,\mathcal{C}_a)$.

I	**II**
x_0^i	
	y_0^i
x_1^i	
	y_1^i
⋮	⋮

Fig. 2.5: Game $G_{cub}(\mathcal{C}_{a_i})$

I	**II**
u_0^i	
	v_0^i
u_1^i	
	v_1^i
⋮	⋮

Fig. 2.6: Game $G_{cub}(\mathcal{C}_{b_i})$

I	**II**
a_0	
	b_0
a_1	
	b_1
⋮	⋮

Fig. 2.7: Game $G_{cub}(\Delta_{a\in A}\,\mathcal{C}_a)$

The idea is as in the previous Lemma, but this time the strategy for player **II** requires to simulate in the background two similar type of games $G_{cub}(\mathcal{C}_{a_i})$ and $G_{cub}(\mathcal{C}_{b_i})$ where the $a_i, b_i \in A$ are the elements that will appear in our main game. Therefore, progressively simulating

$$x_{2j}^i = u_{2j}^i = a_j \text{ and } x_{2j+1}^i = u_{2j+1}^i = b_j \text{ for } j \in \mathbb{N}$$

as the main game advances and using our known strategies as answers will ensure that if player **II** picks her elements by browsing through those answers, the set resulting will be in the diagonal intersection. Therefore, with π being as in Lemma 2.11, we have a winning strategy as follows:

$$b_{2\pi(n,k)} = y_k^n \text{ and } b_{2\pi(n,k)+1} = v_k^n.$$

□

Lemma 2.13. *Let A be a set. For each $a \in A$, take $\mathcal{C}_a \subseteq \mathcal{P}_\omega(A)$. Recall the definition of the diagonal union:*

$$\mathop{\nabla}_{a\in A} \mathcal{C}_a := \left\{ X \in \mathcal{P}_\omega(A) \;\middle|\; X \in \bigcup_{a\in X} \mathcal{C}_a \right\}.$$

*If player **II** has a winning strategy in $G_{cub}(\mathcal{C}_a)$ for some $a \in A$, then she has one in $G_{cub}(\nabla_{a\in A}\,\mathcal{C}_a)$.*

Proof. Let us use the notation of Figure 2.8 for $G_{cub}(\mathcal{C}_a)$, for our fixed $a \in A$, and of Figure 2.9 for $G_{cub}(\nabla_{a\in A}\,\mathcal{C}_a)$.

I	**II**
x_0	
	y_0
x_1	
	y_1
$\vdots$	$\vdots$

Fig. 2.8: Game $G_{cub}(\mathcal{C}_a)$

I	**II**
a_0	
	b_0
a_1	
	b_1
$\vdots$	$\vdots$

Fig. 2.9: Game $G_{cub}(\nabla_{a\in A}\,\mathcal{C}_a)$

It is once again the same idea for the strategy. By simulating

$$x_0 = a \text{ and } x_{i+1} = a_i \text{ for } i \in \mathbb{N},$$

and using the known winning strategy of $G_{cub}(\mathcal{C}_a)$, we have a winning strategy for $G_{cub}(\nabla_{a\in A}\,\mathcal{C}_a)$ as follows:

$$b_0 = a \text{ and } b_{n+1} = y_n \text{ for } n \in \mathbb{N}.$$

□

Definition 2.14. Let $\mathcal{L}$ be a language, $\mathcal{M}$ be an $\mathcal{L}$-structure and s be a variable assignment over $\mathcal{M}$. Take a formula $\phi \in \mathrm{Fml}_{\mathcal{L}}$ and suppose without loss of generality that it is in the *negation normal form*, meaning that it cannot start with a negation symbol $\neg$ unless it is atomic. One can requires this, since every formula is logically equivalent to one of this form because of the logical equivalences that exists between $\vee, \wedge, \rightarrow, \neg$ and between $\forall, \exists, \neg$.

We now define a set $\mathcal{C}_{\phi,s} \subseteq \mathcal{P}_\omega(M)$ by induction on the complexity of ϕ.

- If ϕ is atomic or a negation of an atomic formula, $C_{\phi,s}$ contains all the universes A of countable submodels $\mathcal{A}$ of $\mathcal{M}$ such that $\mathcal{A} \vDash_s \phi$.
- $\mathcal{C}_{\phi\wedge\psi,s} \coloneqq \mathcal{C}_{\phi,s} \cap \mathcal{C}_{\psi,s}$,
- $\mathcal{C}_{\phi\vee\psi,s} \coloneqq \mathcal{C}_{\phi,s} \cup \mathcal{C}_{\psi,s}$,
- $\mathcal{C}_{\forall x\phi,s} \coloneqq \Delta_{a\in M}\,\mathcal{C}_{\phi,s\frac{a}{x}}$.
- $\mathcal{C}_{\exists x\phi,s} \coloneqq \nabla_{a\in M}\,\mathcal{C}_{\phi,s\frac{a}{x}}$,

If $\phi \in \mathrm{Sent}_{\mathcal{L}}$, we write $\mathcal{C}_\phi$ for $\mathcal{C}_{\phi,s}$.

The idea behind this definition is expressed in the following proposition.

Proposition 2.15. *Let $\mathcal{L}$ be a language, $\mathcal{A}$ be an $\mathcal{L}$-structure, $\phi \in \mathrm{Fml}_{\mathcal{L}}$ and s be a variable assignment such that $A \in \mathcal{C}_{\phi,s}$. Then $\mathcal{A} \vDash_s \phi$.*

Proof. We can without loss of generality suppose ϕ to be in the negation normal form. We will prove the result by induction on the complexity of ϕ.

- If ϕ is atomic or the negation of an atomic formula, then it holds by the definition of $\mathcal{C}_{\phi,s}$.
- If ϕ is $\psi \wedge \theta$:

$$\begin{aligned} A \in \mathcal{C}_{\psi,s} \cap \mathcal{C}_{\theta,s} &\Longrightarrow A \in \mathcal{C}_{\psi,s} \text{ and } A \in \mathcal{C}_{\theta,s} \\ &\overset{\text{I.H.}}{\Longrightarrow} \mathcal{A} \vDash_s \psi \text{ and } \mathcal{A} \vDash_s \theta \\ &\Longrightarrow \mathcal{A} \vDash_s \phi. \end{aligned}$$

- If ϕ is $\psi \vee \theta$:

$$\begin{aligned} A \in \mathcal{C}_{\psi,s} \cup \mathcal{C}_{\theta,s} &\Longrightarrow A \in \mathcal{C}_{\psi,s} \text{ or } A \in \mathcal{C}_{\theta,s} \\ &\overset{\text{I.H.}}{\Longrightarrow} \mathcal{A} \vDash_s \psi \text{ or } \mathcal{A} \vDash_s \theta \\ &\Longrightarrow \mathcal{A} \vDash_s \phi. \end{aligned}$$

- If ϕ is $\forall x \psi$:

$$\begin{aligned} A \in \underset{a \in M}{\Delta} \mathcal{C}_{\psi,s\frac{a}{x}} &\Longrightarrow A \in \mathcal{C}_{\psi,s\frac{a}{x}} \text{ for all } a \in A \\ &\overset{\text{I.H.}}{\Longrightarrow} \mathcal{A} \vDash_{s\frac{a}{x}} \psi \text{ for all } a \in A \\ &\Longrightarrow \mathcal{A} \vDash_s \phi. \end{aligned}$$

- If ϕ is $\exists x \psi$:

$$\begin{aligned} A \in \underset{a \in M}{\nabla} \mathcal{C}_{\psi,s\frac{a}{x}} &\Longrightarrow A \in \mathcal{C}_{\psi,s\frac{a}{x}} \text{ for some } a \in A \\ &\overset{\text{I.H.}}{\Longrightarrow} \mathcal{A} \vDash_{s\frac{a}{x}} \psi \text{ for some } a \in A \\ &\Longrightarrow \mathcal{A} \vDash_s \phi. \end{aligned}$$

□

Proposition 2.16. *Let $\mathcal{L}$ be a countable language, $\mathcal{A}$ be an $\mathcal{L}$-structure and $\sigma \in \mathrm{Sent}_{\mathcal{L}}$ such that $\mathcal{A} \vDash \sigma$. Then player **II** has a winning strategy in $G_{cub}(\mathcal{C}_\sigma)$.*

Proof. We will prove it by induction on the complexity of σ.

- If σ is atomic or the negation of an atomic sentence, it follows from Lemma 2.10.
- If σ is $\rho \wedge \tau$, it follows from Lemma 2.11.
- If σ is $\rho \vee \tau$, it is obvious by the I.H. since $\mathcal{C}_\rho \subseteq \mathcal{C}_\sigma$ and $\mathcal{C}_\tau \subseteq \mathcal{C}_\sigma$.
- If σ is $\forall x \phi$, it follows from Lemma 2.12.

- If σ is $\exists x \phi$, it follows from Lemma 2.13.

□

Theorem 2.17 (Löwenheim-Skolem Theorem). *Let $\mathcal{L}$ be a countable language, $\mathcal{M}$ be an $\mathcal{L}$-structure and T be a countable $\mathcal{L}$-theory. If $\mathcal{M} \models T$, then player **II** has a winning strategy in*

$$G_{cub}(\{A \in \mathcal{P}_\omega(M) : \mathcal{A} \textit{ is a countable submodel of } \mathcal{M} \textit{ and } \mathcal{A} \models T\}).$$

In particular, T has a countable model which is a submodel of $\mathcal{M}$.

Proof. Let us enumerate the theory $T = \{\sigma_0, \sigma_1, \ldots\}$. By Proposition 2.16, player **II** has a winning strategy in $G_{cub}(\mathcal{C}_{\sigma_n})$ for all $n \in \mathbb{N}$. By Lemma 2.11, player **II** has a winning strategy in

$$G_{cub}\left(\bigcap_{n\geqslant 0} \mathcal{C}_{\sigma_n}\right) \overset{\text{Proposition 2.15}}{=} G_{cub}(\{A \in \mathcal{P}_\omega(M) : \mathcal{A} \text{ is a countable submodel of } \mathcal{M} \text{ and } \mathcal{A} \models T\}).$$

Therefore, this set must be non-empty and hence T has a countable model which is a submodel of $\mathcal{M}$. □

2.3 Ultrafilters, ultraproducts and compactness theorem

This section is at the same time similar to the last two by giving an opening on a new domain, here filters and ultraproducts, to prove a classical result of logic, but it also dives this time more deeply into the concerned subject, since it will appear again in the next chapter.

Definition 2.18. Let X be a set. A *filter* on X is a subset $\mathcal{F}$ of $\mathcal{P}(X)$ satisfying the following axioms.

1. $\varnothing \notin \mathcal{F}$.
2. $\mathcal{F} \neq \varnothing$.
3. Closure under intersections: Whenever $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$.
4. Closure under supersets: Whenever $A \in \mathcal{F}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{F}$.

Definition 2.19. Let X be a set. A subset $\mathcal{H}$ of $\mathcal{P}(X)$ has the *finite intersection property*, abbreviated *fip*, if it is non-empty and if the intersection of every non-empty finite subset of $\mathcal{H}$ is non-empty, i.e.,

$$\text{for all } C_1, \ldots, C_n \in \mathcal{H} : C_1 \cap \ldots \cap C_n \neq \varnothing.$$

Observe that every filter has the fip, thanks to axioms (1) and (3). Conversely, the presence of a subset having the fip, can be used to create a filter as follows.

Proposition 2.20. *Let X be a set and $\mathcal{H} \subseteq \mathcal{P}(X)$ have the fip. Then*

$$\mathcal{F}^{\mathcal{H}} := \{A \subseteq X \mid \exists C_1, \ldots, C_n \in \mathcal{H} : C_1 \cap \ldots \cap C_n \subseteq A\}$$

is a filter on X, called the filter generated by $\mathcal{H}$.

Proof. Let us check that the axioms for a filter hold:

1. As $\mathcal{H}$ has the fip, finite intersections of elements of $\mathcal{H}$ are never empty and thus $\varnothing \notin \mathcal{F}$.
2. Observe that $\varnothing \neq \mathcal{H} \subseteq \mathcal{F}^{\mathcal{H}}$.
3. Take $A, B \in \mathcal{F}^{\mathcal{H}}$, with $C_1, \ldots, C_n, D_1, \ldots, D_m \in \mathcal{H}$ such that $C_1 \cap \ldots \cap C_n \subseteq A$ and $D_1 \cap \ldots \cap D_m \subseteq B$. Then $C_1 \cap \ldots \cap C_n \cap D_1 \cap \ldots \cap D_m \subseteq A \cap B$, which is therefore in $\mathcal{F}^{\mathcal{H}}$.
4. Take $A \in \mathcal{F}^{\mathcal{H}}$ with $C_1, \ldots, C_n \in \mathcal{H}$ such that $C_1 \cap \ldots \cap C_n \subseteq A$ and take $A \subseteq B \subseteq X$. Then $C_1 \cap \ldots \cap C_n \subseteq B$ and $B \in \mathcal{F}^{\mathcal{H}}$.

□

Definition 2.21. Let X be a set. Here are two definitions of an *ultrafilter*.

(a) An *ultrafilter* $\mathcal{U}$ on X is a filter on X which is maximal under containment, i.e.,

$$\forall \mathcal{F} \text{ filter on } X : \mathcal{U} \subseteq \mathcal{F} \Longrightarrow \mathcal{U} = \mathcal{F}.$$

(b) An *ultrafilter* $\mathcal{U}$ on X is a filter on X which satisfies

$$\forall A \subseteq X : \text{ either } A \in \mathcal{U} \text{ or } X \setminus A \in \mathcal{U}.$$

Proposition 2.22. *The two definitions of ultrafilter are equivalent.*

Proof. Let X be a set and $\mathcal{U}$ be a filter on X.

($a \Rightarrow b$) By contrapositive, suppose there exists $A \subseteq X$, with $A \notin \mathcal{U}$ and $X \setminus A \notin \mathcal{U}$. Observe that $A \neq \varnothing$, as $X \setminus \varnothing = X$ is in all filters. Let us prove that $\mathcal{U} \cup \{A\}$ has the fip by supposing, ab absurdum, that it does not. Since $\mathcal{U}$ has the fip, A must be part of the intersection which ends up being empty, i.e., there exists $C_1, \ldots, C_n \in \mathcal{U}$ such that $A \cap C_1 \cap \ldots \cap C_n = \varnothing$. Let $C := C_1 \cap \ldots \cap C_n$ which is in $\mathcal{U}$ by closure under intersections. Therefore, we have $C \subseteq X \setminus A$ and hence $X \setminus A \in \mathcal{U}$ by closure under supersets, which is a contradiction with our suppositions ↯. Therefore, Proposition 2.20 implies the existence of a filter strictly containing $\mathcal{U}$, which proves the negation of definition (a) as desired.

($b \Rightarrow a$) By contrapositive, suppose that there exists a filter $\mathcal{F}$ on X which strictly contains $\mathcal{U}$. Then there exists some $A \in \mathcal{F} \setminus \mathcal{U}$. Observe that $X \setminus A \notin \mathcal{U}$ as it would otherwise also be in $\mathcal{F}$ and thus $\varnothing = A \cap (X \setminus A) \in \mathcal{F}$. Therefore, A and $X \setminus A$ are both not in $\mathcal{U}$, which proves the negation of definition (b) as desired.

□

Theorem 2.23 (The ultrafilter theorem)**.** *Every filter can be extended to an ultrafilter.*

Proof. Let X be a set and $\mathcal{F}$ be a filter on X Consider the collection

$$Y := \{\mathcal{G} \subseteq \mathcal{P}(X) \mid \mathcal{G} \text{ is a filter on } X \text{ and } \mathcal{F} \subseteq \mathcal{G}\}.$$

As $\mathcal{F} \in Y$, it is non-empty. Take a $\subseteq$-chain $C \subseteq Y$. Observe that $\bigcup C$ has the fip. Indeed, take $A_1, \ldots, A_n \in \bigcup C$. Hence there exists $\mathcal{G}_1, \ldots, \mathcal{G}_n \in C$ such that $A_i \in \mathcal{G}_i$ for $1 \leqslant i \leqslant n$, and without loss of generality $\mathcal{G}_1 \subseteq \ldots \subseteq \mathcal{G}_n$, since they belong in a chain. Then $A_1, \ldots, A_n \in \mathcal{G}_n$, and since $\mathcal{G}_n$ is a filter, we have $A_1 \cap \ldots \cap A_n \neq \varnothing$. Therefore, by Proposition 2.20 we have $\mathcal{F}^{\bigcup C}$ which is a $\subseteq$-maximal element to the chain. By Zorn's lemma, Y has a $\subseteq$-maximal element $\mathcal{U}$, which extends $\mathcal{F}$ and is an ultrafilter by maximality. □

The following notion will not be of use to demonstrate the compactness theorem, but it will be useful in the next chapter.

Definition 2.24. Let I be a set. An ultrafilter $\mathcal{U}$ on I is *regular* if there exists a bijection $f : I \to \mathcal{P}_{<\omega}(I)$ such that for all $i \in I$:

$$\{j \in I : i \in f(j)\} \in \mathcal{U}.$$

Note. Observe that $I \approx \mathcal{P}_{<\omega}(I)$ if and only if I is infinite. Therefore, this notion is interesting only when I is infinite. We even have existence of a regular ultrafilter in this case, as proved in the next lemma.

Lemma 2.25. *Let I be an infinite set. Then there exists a regular ultrafilter on I.*

Proof. Let $f : I \to \mathcal{P}_{<\omega}(I)$ be an arbitrary bijection. For each $i \in I$, let $E_i := \{j \in I : i \in f(j)\}$. We want to prove that the collection of those sets, $E := \{E_i : i \in I\}$ has the fip. Take any $E_{i_1}, \ldots, E_{i_n} \in E$. Let $i_0 := f^{-1}(\{i_1, \ldots, i_n\}) \in I$. Then $i_1, \ldots, i_n \in f(i_0)$ and thus $i_0 \in E_{i_1} \cap \ldots \cap E_{i_n}$ which must be non-empty. Therefore, E has the fip and by Proposition 2.20 and Theorem 2.23, it is contained in an ultrafilter on I which is regular because of its construction. □

Definition 2.26. Let $\mathcal{L}$ be a language, $I \neq \varnothing$ be a set, $(\mathcal{M}_i)_{i \in I}$ be a collection of $\mathcal{L}$-structures and $\mathcal{U}$ be an ultrafilter on I. The *ultraproduct* of $(\mathcal{M}_i)_{i \in I}$ by $\mathcal{U}$ is the $\mathcal{L}$-structure denoted

$$\mathcal{M} := \prod_{i \in I} \mathcal{M}_i /\mathcal{U}$$

and defined as follows.

- To define the universe, we first describe an equivalence relation $\sim$ on $\prod_{i \in I} M_i$ thanks to the ultrafilter $\mathcal{U}$. For $a = (a_i)_{i \in I}, b = (b_i)_{i \in I} \in \prod_{i \in I} M_i$, define

$$a \sim b :\Longleftrightarrow \{i \in I \mid a_i = b_i\} \in \mathcal{U}.$$

- Reflexivity: It is immediate that $a \sim a$ since $\{i \in I \mid a_i = a_i\} = I \in \mathcal{U}$.
- Symmetry: Clear.
- Transitivity: If $c = (c_i)_{i\in I} \in \prod_{i\in I} M_i$ and $a \sim b \sim c$, then $A \coloneqq \{i \in I \mid a_i = b_i\}, B \coloneqq \{i \in I \mid b_i = c_i\} \in \mathcal{U}$. Therefore $\{i \in I \mid a_i = c_i\} \supseteq A \cap B$ and thus is in $\mathcal{U}$.

For an element $a \in \prod_{i\in I} M_i$, its equivalent class will be denoted by $[a]$. We can now define the universe of the ultraproduct as

$$M \coloneqq \prod_{i\in I} M_i \,/\!\sim.$$

- For a constant symbol $c \in \mathcal{L}$, let

$$c^{\mathcal{M}} \coloneqq \left[\left(c^{\mathcal{M}_i}\right)_{i\in I}\right].$$

- For an n-ary function symbol $f \in \mathcal{L}$ and $a^1, \ldots, a^n \in \prod_{i\in I} M_i$, let

$$f^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right) \coloneqq \left[\left(f^{\mathcal{M}_i}(a^1_i, \ldots, a^n_i)\right)_{i\in I}\right].$$

- For an n-ary predicate symbol $R \in \mathcal{L}$ and $a^1, \ldots, a^n \in \prod_{i\in I} M_i$, let

$$\left(\left[a^1\right], \ldots, [a^n]\right) \in R^{\mathcal{M}} :\Longleftrightarrow \left\{i \in I \;\middle|\; (a^1_i, \ldots, a^n_i) \in R^{\mathcal{M}_i}\right\} \in \mathcal{U}.$$

Let us check that everything is well-defined. For that, let $a^1, \ldots, a^n, b^1, \ldots, b^n, c \in \prod_{i\in I} M_i$. First, suppose $R \in \mathcal{L}$ is an n-ary predicate symbol, $\left(\left[a^1\right], \ldots, [a^n]\right) \in R^{\mathcal{M}}$ and $a^j \sim b^j$ for $1 \leqslant j \leqslant n$. Observe that

$$\begin{aligned}\left\{i \in I \mid (b^1_i, \ldots, b^n_i) \in R^{\mathcal{M}_i}\right\} &\supseteq \left\{i \in I \;\middle|\; \text{for all } 1 \leqslant j \leqslant n : a^j_i = b^j_i \text{ and } (a^1_i, \ldots, a^n_i) \in R^{\mathcal{M}}\right\} \\ &= \bigcap_{j=1}^{n} \left\{i \in I \;\middle|\; a^j_i = b^j_i\right\} \cap \left\{i \in I \mid (a^1_i, \ldots, a^n_i) \in R^{\mathcal{M}_i}\right\}.\end{aligned}$$

By closure under intersection and supersets, we have that $\{i \in I \mid (b^1_i, \ldots, b^n_i) \in R^{\mathcal{M}}\} \in \mathcal{U}$, or in other words that $\left(\left[b^1\right], \ldots, [b^n]\right) \in R^{\mathcal{M}}$ too.

Now, suppose $f \in \mathcal{L}$ is an n-ary function symbol, $f^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right) = [c]$ and $a^j \sim b^j$ for $1 \leqslant j \leqslant n$. Observe that

$$\begin{aligned}\left\{i \in I \mid f^{\mathcal{M}_i}(b^1_i, \ldots, b^n_i) = c_i\right\} &\supseteq \left\{i \in I \;\middle|\; \text{for all } 1 \leqslant j \leqslant n : a^j_i = b^j_i \text{ and } f^{\mathcal{M}_i}(a^1_i, \ldots, a^n_i) = c_i\right\} \\ &= \bigcap_{j=1}^{n} \left\{i \in I \;\middle|\; a^j_i = b^j_i\right\} \cap \left\{i \in I \mid f^{\mathcal{M}_i}(a^1_i, \ldots, a^n_i) = c_i\right\}.\end{aligned}$$

By closure under intersection and supersets, we have that $\{i \in I \mid f^{\mathcal{M}_i}(b_i^1, \dots, b_i^n) = c_i\} \in \mathcal{U}$. Hence

$$f^{\mathcal{M}}\left(\left[b^1\right], \dots, \left[b^n\right]\right) = \left[\left(f^{\mathcal{M}_i}(b_i^1, \dots, b_i^n)\right)_{i \in I}\right] = [c] = f^{\mathcal{M}}\left(\left[a^1\right], \dots, \left[a^n\right]\right).$$

If we take our collection of $\mathcal{L}$-structures to always be the same, that is $\mathcal{M}_i = \mathcal{N}$ for all $i \in I$, the ultraproduct is then called an *ultrapower* and denoted ${}^I\mathcal{N} / \mathcal{U}$.

Here come some observations about the cardinality of ultraproducts, which will be of use in the next chapter.

Lemma 2.27. *Let I be a set, $\{A_i : i \in I\}, \{B_i : i \in I\}$ be collections of sets and $\mathcal{U}$ be an ultrafilter on I. If* $\operatorname{card}(A_i) = \operatorname{card}(B_i)$ *for each $i \in I$, then*

$$\operatorname{card}\left(\prod_{i \in I} \mathcal{A}_i / \mathcal{U}\right) = \operatorname{card}\left(\prod_{i \in I} \mathcal{B}_i / \mathcal{U}\right)$$

Proof. Let $f_i : A_i \approx B_i$ be a bijection for each $i \in I$. We define

$$f : \prod_{i \in I} A_i \to \prod_{i \in I} B_i : (a_i)_{i \in I} \mapsto (f_i(a_i))_{i \in I},$$
$$f^* : \prod_{i \in I} \mathcal{A}_i / \mathcal{U} \to \prod_{i \in I} \mathcal{B}_i / \mathcal{U} : [a] \mapsto [f(a)].$$

Well-defined and injectivity: Take $a = (a_i)_{i \in I}, b = (b_i)_{i \in I} \in \prod_{i \in I} A_i$. Observe that

$$\begin{aligned} a \sim b &\implies \{i \in I : a_i = b_i\} \in \mathcal{U} \\ &\implies \{i \in I : f_i(a_i) = f_i(b_i)\} = \{i \in I : a_i = b_i\} \in \mathcal{U} \\ &\implies f(a) \sim f(b) \\ &\implies f^*([a]) = f^*([b]). \end{aligned}$$

Surjectivity: Take $b = (b_i)_{i \in I} \in \prod_{i \in I} B_i$. Observe that it is the image through f^* of $\left[\left(f_i^{-1}(b_i)\right)_{i \in I}\right]$. ☐

Lemma 2.28. *Let $n < \omega$, I be a non-empty set and $\mathcal{U}$ be an ultrafilter on I. Then*

$$\operatorname{card}\left({}^I n / \mathcal{U}\right) = n.$$

Proof. If n is 0 or 1, it is trivial. Otherwise, let

$$f : n \to {}^I n / \mathcal{U} : m \mapsto [g_m : I \to n : i \mapsto m].$$

Injectivity: Take $m < k < n$. Observe that

$$\{i \in I \mid g_m(i) = g_k(i)\} = \{i \in I \mid m = k\} = \varnothing \notin \mathcal{U}.$$

Hence, we have $[g_m] \neq [g_k]$, i.e., f is injective.

Surjectivity: Take $h : I \to n$. We want to show that $[h]$ is in the image of f. Let $m_1, \dots, m_k < n$ be distinct such that $\{h^{-1}(m_1), \dots, h^{-1}(m_k)\}$ forms a partition of I. Suppose ab absurdum that $h^{-1}(m_j) \notin \mathcal{U}$ for all $1 \leqslant j \leqslant k$. Then, by a quick induction we see that

$$\begin{aligned} h^{-1}(m_1) \notin \mathcal{U} &\Longrightarrow I \setminus h^{-1}(m_1) \in \mathcal{U} \\ h^{-1}(m_2) \notin \mathcal{U} &\Longrightarrow I \setminus (h^{-1}(m_1) \cup h^{-1}(m_2)) \in \mathcal{U} \\ \dots \quad & \quad \dots \\ h^{-1}(m_k) \notin \mathcal{U} &\Longrightarrow I \setminus I = \varnothing \in \mathcal{U} \ \lightning. \end{aligned}$$

Therefore, there exists some $1 \leqslant j \leqslant k$ such that $h^{-1}(m_j) \in \mathcal{U}$ Hence

$$\left\{ i \in I \;\middle|\; h(i) = g_{m_j}(i) = m_j \right\} = h^{-1}(m_j) \in \mathcal{U},$$

or in other words, $[h] = \left[g_{m_j} \right] = f(m_j)$, which proves that f is surjective.

We have shown that f is a bijection, which concludes the proof. □

Lemma 2.29. *Let I be a set, $\{\lambda_i : i \in I\}$ be a collection of cardinal numbers and $\mathcal{U}$ be an ultrafilter on I. Then*

$$\operatorname{card}\left(\prod_{i \in I} \lambda_i \,/\mathcal{U} \right) \leqslant \operatorname{card}\left(\prod_{i \in I} \lambda_i \right)$$

Proof. Let

$$f : \prod_{i \in I} \lambda_i \,/\mathcal{U} \to \prod_{i \in I} \lambda_i : [a] \mapsto a,$$

where a above is one fixed representative of the equivalence class which has been arbitrarily chosen. This is clearly an injection, which proves the lemma. □

Corollary 2.30. *Let λ be a cardinal, I be a set of cardinality μ and $\mathcal{U}$ be an ultrafilter on I. Then*

$$\operatorname{card}\left({}^{I}\lambda \,/\, \mathcal{U} \right) \leqslant \lambda^{\mu}.$$

Theorem 2.31. *Let λ be a cardinal, I be a set of cardinality μ and $\mathcal{U}$ be a regular ultrafilter on I. Then*

$$\operatorname{card}\left({}^{I}\lambda \,/\, \mathcal{U} \right) = \lambda^{\mu}.$$

Proof. The first inequality comes from Corollary 2.30, so there is only the other one to prove. Let A be the set of finite sequences of element of λ, i.e., $A = \lambda^{<\omega}$. Since λ is infinite, we have $\operatorname{card}(A) = \lambda$, and thus by Lemma 2.27, we have that ${}^{I}A \,/\, \mathcal{U} \approx {}^{I}\lambda \,/\, \mathcal{U}$. Therefore, it is enough to prove that ${}^{I}\lambda \preccurlyeq {}^{I}A \,/\, \mathcal{U}$.

By hypothesis, $\mathcal{U}$ is regular, hence we have a bijection $f : I \to \mathcal{P}_{<\omega}(I)$ such that for each $i \in I$, $E_i \coloneqq \{j \in I : i \in f(j)\} \in \mathcal{U}$. Take $g \in {}^{I}\lambda$. We want to send it to an element of ${}^{I}A \,/\, \mathcal{U}$. For that, let us define an element $g^* \in {}^I A$. For each $i \in I$, $f(i) \in \mathcal{P}_{<\omega}(I)$, say $f(i) = \{i_0, \dots, i_k\}$, and thus let $g^*(i) = \langle g(i_0), \dots, g(i_k) \rangle$. We can now define

$$\pi : {}^I\lambda \to {}^I A \,/\, \mathcal{U} : g \mapsto [g^*] .$$

Injectivity: Take $g, h \in {}^I\lambda$. Observe that

$$\begin{aligned} g \neq h &\Longrightarrow \exists i \in I : g(i) \neq h(i) \\ &\Longrightarrow \text{for each } j \in I \text{ with } i \in f(j), \text{ we have } g^*(j) \neq h^*(j) \\ &\Longrightarrow \{j \in I : g^*(j) \neq h^*(j)\} \supseteq \{j \in I : i \in f(j)\} = E_i \in \mathcal{U} \\ &\overset{\text{closure supersets}}{\Longrightarrow} \{j \in I : g^*(j) \neq h^*(j)\} \in \mathcal{U} \\ &\overset{\text{maximality}}{\Longrightarrow} \{j \in I : g^*(j) = h^*(j)\} \notin \mathcal{U} \\ &\Longrightarrow [g^*] \neq [h^*] . \end{aligned}$$

Therefore, we have ${}^I\lambda \preccurlyeq {}^I A \,/\, \mathcal{U} \approx {}^I\lambda \,/\, \mathcal{U}$, which completes the proof. □

Now comes the most important result about ultraproducts.

Theorem 2.32 (Łoš's Theorem)**.** *Let $\mathcal{L}$ be a language, $I \neq \varnothing$ be a set, $(\mathcal{M}_i)_{i\in I}$ be a collection of $\mathcal{L}$-structures, $\mathcal{U}$ be an ultrafilter on I, $\phi(x_1, \dots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a^1, \dots, a^n \in \prod_{i\in I} \mathcal{M}_i$ with $a^j = (a^j_i)_{i\in I}$ for $1 \leqslant j \leqslant n$. Then*

$$\prod_{i\in I} \mathcal{M}_i \,/\mathcal{U} \vDash \phi\left(\left[a^1\right], \dots, [a^n]\right) \Longleftrightarrow \left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \phi(a^1_i, \dots, a^n_i)\right\} \in \mathcal{U}.$$

Proof. We will denote the ultraproduct by

$$\mathcal{M} := \prod_{i\in I} \mathcal{M}_i \,/\mathcal{U}.$$

Let $t(x_1, \dots, x_n) \in \mathrm{Tm}_{\mathcal{L}}$. We will prove by induction on its complexity that the following assertion holds.

Claim: $t^{\mathcal{M}}\left(\left[a^1\right], \dots, [a^n]\right) = \left[\left(t^{\mathcal{M}_i}(a^1_i, \dots, a^n_i)\right)_{i\in I}\right]$.

- If t is x_j for some $1 \leqslant j \leqslant n$:

$$t^{\mathcal{M}}\left(\left[a^1\right], \dots, [a^n]\right) = \left[a^j\right] = \left[\left(a^j_i\right)_{i\in I}\right] = \left[\left(t^{\mathcal{M}_i}(a^1_i, \dots, a^n_i)\right)_{i\in I}\right].$$

- If t is c where $c \in \mathcal{L}$ is a constant symbol:

$$t^{\mathcal{M}}\left(\left[a^1\right], \dots, [a^n]\right) = c^{\mathcal{M}} = \left[\left(c^{\mathcal{M}_i}\right)_{i\in I}\right] = \left[\left(t^{\mathcal{M}_i}(a^1_i, \dots, a^n_i)\right)_{i\in I}\right].$$

- If t is $f(t_1, \ldots, t_k)$ where $f \in \mathcal{L}$ is an k-ary function symbol:

$$
\begin{aligned}
t^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right) &= f^{\mathcal{M}}\Big(t_1^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right), \ldots, t_k^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right)\Big) \\
&\overset{\text{I.H.}}{=} f^{\mathcal{M}}\left(\left[\left(t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I}\right], \ldots, \left[\left(t_k^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I}\right]\right) \\
&\overset{\text{Def.}}{=} \left[\left(f^{\mathcal{M}_i}\left(t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n), \ldots, t_k^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)\right)_{i\in I}\right] \\
&= \left[\left(t^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I}\right].
\end{aligned}
$$

Lets us now prove the theorem by induction on the complexity of ϕ.

- If ϕ is $t_1 = t_2$:

$$
\begin{aligned}
\mathcal{M} \vDash \phi\left(\left[a^1\right], \ldots, [a^n]\right) &\iff t_1^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right) = t_2^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right) \\
&\overset{\text{Claim}}{\iff} \left(t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I} \sim \left(t_2^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I} \\
&\iff \left\{i \in I \;\middle|\; t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n) = t_2^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U} \\
&\iff \left\{i \in I \mid \mathcal{M}_i \vDash \phi(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U}.
\end{aligned}
$$

- If ϕ is $R(t_1, \ldots, t_m)$, where $R \in \mathcal{L}$ is an m-ary predicate symbol:

$$
\begin{aligned}
\mathcal{M} \vDash \phi\left(\left[a^1\right], \ldots, [a^n]\right) &\iff \Big(t_1^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right), \ldots, t_m^{\mathcal{M}}\left(\left[a^1\right], \ldots, [a^n]\right)\Big) \in R^{\mathcal{M}} \\
&\overset{\text{Claim}}{\iff} \left(\left[\left(t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I}\right], \ldots, \left[\left(t_m^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right)_{i\in I}\right]\right) \in R^{\mathcal{M}} \\
&\overset{\text{Def.}}{\iff} \left\{i \in I \;\middle|\; \left(t_1^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n), \ldots, t_m^{\mathcal{M}_i}(a_i^1, \ldots, a_i^n)\right) \in R^{\mathcal{M}_i}\right\} \in \mathcal{U} \\
&\iff \left\{i \in I \mid \mathcal{M}_i \vDash \phi(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U}.
\end{aligned}
$$

- If ϕ is $\neg\psi$:

$$
\begin{aligned}
\mathcal{M} \vDash \phi\left(\left[a^1\right], \ldots, [a^n]\right) &\iff \mathcal{M} \nvDash \psi\left(\left[a^1\right], \ldots, [a^n]\right) \\
&\overset{\text{I.H.}}{\iff} \left\{i \in I \mid \mathcal{M}_i \vDash \psi(a_i^1, \ldots, a_i^n)\right\} \notin \mathcal{U} \\
&\overset{\text{Maximality}}{\iff} \left\{i \in I \mid \mathcal{M}_i \nvDash \psi(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U} \\
&\iff \left\{i \in I \mid \mathcal{M}_i \vDash \phi(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U}.
\end{aligned}
$$

- If ϕ is $\psi \wedge \theta$:

$$
\begin{aligned}
\mathcal{M} \vDash \phi\left(\left[a^1\right], \ldots, [a^n]\right) &\overset{\text{I.H.}}{\iff} \left\{i \in I \mid \mathcal{M}_i \vDash \psi(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U} \\
&\qquad \text{and } \left\{i \in I \mid \mathcal{M}_i \vDash \theta(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U} \\
&\overset{\text{Closure axioms}}{\iff} \left\{i \in I \mid \mathcal{M}_i \vDash \psi(a_i^1, \ldots, a_i^n) \text{ and } \mathcal{M}_i \vDash \theta(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U} \\
&\iff \left\{i \in I \mid \mathcal{M}_i \vDash \phi(a_i^1, \ldots, a_i^n)\right\} \in \mathcal{U}.
\end{aligned}
$$

- If ϕ is $\exists x\psi(x, x_1, \dots, x_n)$:

$$\begin{aligned}\mathcal{M} \vDash \phi\left(\left[a^1\right], \dots, \left[a^n\right]\right) &\iff \text{there is some } b \in \prod_{i\in I} M_i : \mathcal{M} \vDash \psi\left([b], \left[a^1\right], \dots, \left[a^n\right]\right) \\ &\overset{\text{I.H.}}{\iff} \text{there is some } b \in \prod_{i\in I} M_i : \left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \psi(b_i, a_i^1, \dots, a_i^n)\right\} \in \mathcal{U} \\ &\overset{\text{Closure supersets}}{\implies} \left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \exists x\psi(a_i^1, \dots, a_i^n)\right\} \in \mathcal{U},\end{aligned}$$

since $\left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \psi(b_i, a_i^1, \dots, a_i^n)\right\} \subseteq \left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \exists x\psi(a_i^1, \dots, a_i^n)\right\}$.

For the converse implication, suppose that $J \coloneqq \left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \exists x\psi(a_i^1, \dots, a_i^n)\right\}$ belongs to $\mathcal{U}$. For all $i \in J$, let $c_i \in M_i$ be such that $\mathcal{M}_i \vDash \psi(c_i, a_i^1, \dots, a_i^n)$. As for $i \in I \setminus J$, let c_i be an arbitrary element of M_i. Then, by construction:

$$\left\{i \in I \;\middle|\; \mathcal{M}_i \vDash \psi(c_i, a_i^1, \dots, a_i^n)\right\} \supseteq J, \text{ and } J \in \mathcal{U}.$$

Since $[c] = \left[(c)_{i\in I}\right]$ is such that $\mathcal{M} \vDash \psi\left([c], \left[a^1\right], \dots, \left[a^n\right]\right)$ then $\mathcal{M} \vDash \phi\left(\left[a^1\right], \dots, \left[a^n\right]\right)$, which finishes the proof.

□

When considering theories, this theorem gives an immediate corollary.

Corollary 2.33. *Let $\mathcal{L}$ be a language, $I \neq \varnothing$ be a set, $\mathcal{M}$ be an $\mathcal{L}$-structure, $\mathcal{U}$ be an ultrafilter on I and T be an $\mathcal{L}$-theory. Then,*

$$^I\mathcal{M} / \mathcal{U} \vDash T \iff \mathcal{M} \vDash T.$$

We have now everything to prove the compactness theorem.

Theorem 2.34 (Compactness theorem)**.** *Let $\mathcal{L}$ be a language and T be an $\mathcal{L}$-theory. If every finite $T_0 \subseteq T$ is satisfiable, then T is also satisfiable.*

Proof. Observe that if $T = \varnothing$, then it is trivially satisfiable. Hence, suppose T to be non-empty. Let $I = \mathcal{P}_{<\omega}(T)$, which must also be non-empty. For each $i \in I$, there exists by hypothesis a model $\mathcal{M}_i$ of i. We want to define an ultrafilter $\mathcal{U}$ on I in order to construct an ultraproduct. For each $j \in I$, let $A_j \coloneqq \{i \in I \mid \mathcal{M}_i \vDash j\}$, which is non-empty since $j \in A_j$. We claim that $B \coloneqq \{A_j \mid j \in I\}$ has the fip. Indeed, for $j_1, \dots, j_n \in I$:

$$\begin{aligned}A_{j_1} \cap \dots \cap A_{j_n} &= \{i \in I \mid \mathcal{M}_i \vDash j_k \text{ for } 1 \leqslant k \leqslant n\} \\ &= \{i \in I \mid \mathcal{M}_i \vDash (j_1 \cup \dots \cup j_n)\} \\ &= A_{j_1 \cup \dots \cup j_n} \neq \varnothing.\end{aligned}$$

Therefore, by Proposition 2.20 and Theorem 2.23, B is contained in an ultrafilter $\mathcal{U}$. For each $\sigma \in T$, Łoš's Theorem implies that

$$\{i \in I \mid \mathcal{M}_i \vDash \sigma\} = A_{\{\sigma\}} \in \mathcal{U} \Longrightarrow \prod_{i \in I} \mathcal{M}_i \,/\mathcal{U} \vDash \sigma.$$

This prove that the ultraproduct of $(\mathcal{M}_i)_{i \in I}$ by $\mathcal{U}$ is a model of T, which finishes the proof.

□

Chapter 3

Homogeneous, Saturated, Special Models and Vaught Theorem

3.1 Model theory introduction

Here is a recall of all the basic notions of model theory, summarized in one definition.

Definition 3.1. Let $\mathcal{L}$ be a language and $\mathcal{M}, \mathcal{N}$ be two $\mathcal{L}$-structures.

- An ($\mathcal{L}$-)*homomorphism* H from $\mathcal{M}$ to $\mathcal{N}$, denoted $H : \mathcal{M} \to \mathcal{N}$, is a function $M \to N$ preserving the interpretation of constant, function and predicate symbols. More precisely:
    1. For each constant symbol $c \in \mathcal{L}$: $H(c^{\mathcal{M}}) = c^{\mathcal{N}}$.
    2. For each n-ary function symbol $f \in \mathcal{L}$ and each $a_1, \ldots, a_n \in M$:
    $$H\left(f^{\mathcal{M}}(a_1, \ldots, a_n)\right) = f^{\mathcal{N}}\left(H(a_1), \ldots, H(a_n)\right).$$
    3. For each n-ary predicate symbol $R \in \mathcal{L}$ and each $a_1, \ldots, a_n \in M$:
    $$(a_1, \ldots, a_n) \in R^{\mathcal{M}} \Longrightarrow (H(a_1), \ldots, H(a_n)) \in R^{\mathcal{N}}.$$
- An ($\mathcal{L}$-)homomorphism $H : \mathcal{M} \to \mathcal{N}$ is an ($\mathcal{L}$-)*embedding* from $\mathcal{M}$ to $\mathcal{N}$ if it is injective and for each n-ary predicate symbol $R \in \mathcal{L}$ and each $a_1, \ldots, a_n \in M$:
$$(a_1, \ldots, a_n) \in R^{\mathcal{M}} \Longleftrightarrow (H(a_1), \ldots, H(a_n)) \in R^{\mathcal{N}}.$$
- An *isomorphism* H from $\mathcal{M}$ to $\mathcal{N}$, denoted $H : \mathcal{M} \cong \mathcal{N}$, is a surjective embedding from $\mathcal{M}$ to $\mathcal{N}$.
- An isomorphism H from $\mathcal{M}$ to itself is an *automorphism* of $\mathcal{M}$.

- $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ or $\mathcal{N}$ is an *extension* of $\mathcal{M}$, denoted $\mathcal{M} \subseteq \mathcal{N}$, if $M \subseteq N$ and the inclusion $i : M \to N$ is an embedding. In other words:
    1. For each constant symbol $c \in \mathcal{L}$: $c^{\mathcal{M}} = c^{\mathcal{N}} \in M$.
    2. For each n-ary function symbol $f \in \mathcal{L}$: $f^{\mathcal{M}} = f^{\mathcal{N}}|_{M^n}$.
    3. For each n-ary predicate symbol $R \in \mathcal{L}$: $R^{\mathcal{M}} = R^{\mathcal{N}} \cap M^n$.
- For a subset $C \subseteq M$, the *restriction* $\mathcal{M}|_C$, or the substructure *generated* by C, is the substructure of $\mathcal{M}$ with universe being the closure of C by the interpretations of function symbols. Formally, let

$$\begin{aligned} C_0 &\coloneqq C \\ C_{n+1} &\coloneqq \left\{ f^{\mathcal{M}}(a_1, \ldots, a_k) : f \in \mathcal{L} \text{ is a } k\text{-ary predicate symbol and } a_1, \ldots, a_k \in C_n \right\}, \end{aligned}$$

  where k as above can be 0 because we can consider the constant symbols as being 0-ary function symbols. The universe is then defined as $\bigcup_{n \in \mathbb{N}} C_n$ and each symbol of $\mathcal{L}$ has the same interpretation as in $\mathcal{M}$.
- A substructure of $\mathcal{M}$ is *finitely generated* if it generated by a finite subset $C \subseteq M$.
- $\mathcal{M}$ is ($\mathcal{L}$-)*elementary equivalent* to $\mathcal{N}$, denoted $\mathcal{M} \equiv_{\mathcal{L}} \mathcal{N}$, if for all $\sigma \in \mathrm{Sent}_{\mathcal{L}}$:

$$\mathcal{M} \vDash \sigma \iff \mathcal{N} \vDash \sigma.$$

  When it is clear about which language is considered, the subscript can be omitted.
- An ($\mathcal{L}$-)embedding $H : \mathcal{M} \to \mathcal{N}$ is an ($\mathcal{L}$-)*elementary embedding* of $\mathcal{M}$ into $\mathcal{N}$ if for each formula $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and each $a_1, \ldots, a_n \in M$:

$$\mathcal{M} \vDash \phi(a_1, \ldots, a_n) \iff \mathcal{N} \vDash \phi(H(a_1), \ldots, H(a_n)).$$

  If such an H exists, $\mathcal{M}$ is said to be *elementarily embeddable* in $\mathcal{N}$.
- $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$ or $\mathcal{N}$ is an *elementary extension* of $\mathcal{M}$, denoted $\mathcal{M} \prec \mathcal{N}$, if $\mathcal{M} \subseteq \mathcal{N}$ and the inclusion $i : M \to N$ is an elementary embedding. In other words, for each formula $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and each $a_1, \ldots, a_n \in M$:

$$\mathcal{M} \vDash \phi(a_1, \ldots, a_n) \iff \mathcal{N} \vDash \phi(a_1, \ldots, a_n).$$

Observe that being isomorphic means that symbols are agreeing on their meaning, while being elementary embeddable means that the formulae are agreeing on their meaning. Therefore, this motivates the following proposition.

Proposition 3.2. *Every isomorphism is an elementary embedding.*

Proof. Let $\mathcal{L}$ be a language, $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures and $H : \mathcal{A} \cong \mathcal{B}$ be an isomorphism. First, let $t(x_1, \ldots, x_n) \in \mathrm{Tm}_{\mathcal{L}}$ and $a_1, \ldots, a_n \in A$. We will prove the following claim by induction on the complexity of t:

Claim: $H(t^{\mathcal{A}}(a_1, \ldots, a_n)) = t^{\mathcal{B}}(H(a_1), \ldots, H(a_n))$.

- If t is x_i for some $1 \leqslant i \leqslant n$, then
$$H(t^{\mathcal{A}}(a_1, \ldots, a_n)) = H(a_i) = t^{\mathcal{B}}(H(a_1), \ldots, H(a_n)).$$
- If t is c where $c \in \mathcal{L}$ is a constant symbol, then
$$H(t^{\mathcal{A}}(a_1, \ldots, a_n)) = H(c^{\mathcal{A}}) = c^{\mathcal{B}} = t^{\mathcal{B}}(H(a_1), \ldots, H(a_n)).$$
- If t is $f(t_1, \ldots, t_k)$ where $f \in \mathcal{L}$ is an k-ary function symbol, then
$$\begin{aligned} H(t^{\mathcal{A}}(a_1, \ldots, a_n)) &= H(f^{\mathcal{A}}(t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathcal{A}}(a_1, \ldots, a_n))) \\ &\overset{\text{H isom.}}{=} f^{\mathcal{B}}(H(t_1^{\mathcal{A}}(a_1, \ldots, a_n)), \ldots, H(t_k^{\mathcal{A}}(a_1, \ldots, a_n))) \\ &\overset{\text{I.H.}}{=} f^{\mathcal{B}}(t_1^{\mathcal{B}}(H(a_1), \ldots, H(a_n)), \ldots, t_k^{\mathcal{B}}(H(a_1), \ldots, H(a_n))) \\ &= t^{\mathcal{B}}(H(a_1), \ldots, H(a_n)). \end{aligned}$$

Now, take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a_1, \ldots, a_n \in A$. Let us prove that H is an elementary embedding by induction on the complexity of ϕ.

- If ϕ is $t_1 = t_2$:
$$\begin{aligned} \mathcal{A} \vDash \phi(a_1, \ldots, a_n) &\iff t_1^{\mathcal{A}}(a_1, \ldots, a_n) = t_2^{\mathcal{A}}(a_1, \ldots, a_n) \\ &\overset{\text{H isom.}}{\iff} H(t_1^{\mathcal{A}}(a_1, \ldots, a_n)) = H(t_2^{\mathcal{A}}(a_1, \ldots, a_n)) \\ &\overset{\text{Claim}}{\iff} t_1^{\mathcal{B}}(H(a_1), \ldots, H(a_n)) = t_2^{\mathcal{B}}(H(a_1), \ldots, H(a_n)) \\ &\iff \mathcal{B} \vDash \phi(H(a_1, \ldots, H(a_n)). \end{aligned}$$
- If ϕ is $R(t_1, \ldots, t_k)$:
$$\begin{aligned} \mathcal{A} \vDash \phi(a_1, \ldots, a_n) &\iff (t_1(a_1, \ldots, a_n), \ldots, t_k(a_1, \ldots, a_n)) \in R^{\mathcal{A}} \\ &\overset{\text{H isom.}}{\iff} (H(t_1^{\mathcal{A}}(a_1, \ldots, a_n)), \ldots, H(t_k^{\mathcal{A}}(a_1, \ldots, a_n))) \in R^{\mathcal{B}} \\ &\overset{\text{Claim}}{\iff} (t_1^{\mathcal{B}}(H(a_1), \ldots, H(a_n)), \ldots, t^{\mathcal{B}}(H(a_1), \ldots, H(a_n))) \in R^{\mathcal{B}} \\ &\iff \mathcal{B} \vDash \phi(H(a_1), \ldots, H(a_n)). \end{aligned}$$
- If ϕ is $\neg\psi$:
$$\begin{aligned} \mathcal{A} \vDash \phi(a_1, \ldots, a_n) &\iff \mathcal{A} \nvDash \psi(a_1, \ldots, a_n) \\ &\overset{\text{I.H.}}{\iff} \mathcal{B} \nvDash \psi(H(a_1), \ldots, H(a_n)) \\ &\iff \mathcal{B} \vDash \phi(H(a_1), \ldots, H(a_n)). \end{aligned}$$

- If ϕ is $\psi \wedge \theta$:

$$\begin{aligned}\mathcal{A} \vDash \phi(a_1, \ldots, a_n) &\iff \mathcal{A} \vDash \psi(a_1, \ldots, a_n) \text{ and } \mathcal{A} \vDash \theta(a_1, \ldots, a_n) \\ &\overset{\text{I.H.}}{\iff} \mathcal{B} \vDash \psi(H(a_1), \ldots, H(a_n)) \text{ and } \mathcal{B} \vDash \theta(H(a_1), \ldots, H(a_n)) \\ &\iff \mathcal{B} \vDash \phi(H(a_1), \ldots, H(a_n)).\end{aligned}$$

- If ϕ is $\exists x \psi(x, x_1, \ldots, x_n)$:

$$\begin{aligned}\mathcal{A} \vDash \phi(a_1, \ldots, a_n) &\iff \text{for some } a \in A : \mathcal{A} \vDash \psi(a, a_1, \ldots, a_n) \\ &\overset{\text{I.H. and } a=H(b)}{\iff} \text{for some } b \in B : \mathcal{B} \vDash \psi(b, H(a_1), \ldots, H(a_n)) \\ &\iff \mathcal{B} \vDash \phi(H(a_1), \ldots, H(a_n)).\end{aligned}$$

□

Lemma 3.3. *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be two $\mathcal{L}$-structures with $\mathcal{A} \subseteq \mathcal{B}$. Then*

$$\mathcal{A} \prec \mathcal{B} \iff \forall \phi \in \mathrm{Fml}_{\mathcal{L}} \, \forall s \in {}^{\mathrm{Var}}A : \Big(\mathcal{B} \vDash_s \exists x \phi \Rightarrow \mathcal{B} \vDash_{s\frac{a}{x}} \phi \textit{ for some } a \in A\Big)$$

Proof. ($\Rightarrow$) Let $\phi \in \mathrm{Fml}_{\mathcal{L}}$ and s be a variable assignment over A. We have

$$\begin{aligned}\mathcal{B} \vDash_s \exists x \phi &\overset{\mathcal{A} \prec \mathcal{B}}{\Longrightarrow} \mathcal{A} \vDash_s \exists x \phi \\ &\Longrightarrow \mathcal{A} \vDash_{s\frac{a}{x}} \phi \text{ for some } a \in A \\ &\overset{\mathcal{A} \prec \mathcal{B}}{\Longrightarrow} \mathcal{B} \vDash_{s\frac{a}{x}} \phi \text{ for some } a \in A\end{aligned}$$

($\Leftarrow$) Let $\phi \in \mathrm{Fml}_{\mathcal{L}}$ and s be a variable assignment over A. We will prove $\mathcal{A} \vDash_s \phi \iff \mathcal{B} \vDash_s \phi$ by induction on the complexity of ϕ.

- If ϕ is $t_1 = t_2$:

$$\mathcal{A} \vDash_s \phi \iff t_1(s) = t_2(s) \iff \mathcal{B} \vDash_s \phi$$

- If ϕ is $R(t_1, \ldots, t_n)$:

$$\mathcal{A} \vDash_s \phi \iff (t_1(s), \ldots, t_n(s)) \in R^{\mathcal{A}} = R^{\mathcal{B}} \cap A^n \iff \mathcal{B} \vDash_s \phi$$

- If ϕ is $\neg\psi$:

$$\mathcal{A} \vDash_s \phi \iff \mathcal{A} \nvDash_s \psi \overset{\text{I.H.}}{\iff} \mathcal{B} \nvDash_s \psi \iff \mathcal{B} \vDash_s \phi$$

- If ϕ is $\psi \wedge \theta$:

$$\mathcal{A} \vDash_s \phi \iff \mathcal{A} \vDash_s \psi \text{ and } \mathcal{A} \vDash_s \theta \overset{\text{I.H.}}{\iff} \mathcal{B} \vDash_s \psi \text{ and } \mathcal{B} \vDash_s \theta \iff \mathcal{B} \vDash_s \phi$$

- If ϕ is $\exists x\psi$:

$$\begin{aligned}\mathcal{A} \models_s \phi &\iff \mathcal{A} \models_{s\frac{a}{x}} \psi \text{ for some } a \in A \\ &\overset{\text{I.H.}}{\iff} \mathcal{B} \models_{s\frac{a}{x}} \psi \text{ for some } a \in A \\ &\overset{\text{Hypothesis}}{\iff} \mathcal{B} \models_s \phi\end{aligned}$$

□

An immediate reformulation of this lemma is as follows.

Corollary 3.4. *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be two $\mathcal{L}$-structures with $\mathcal{A} \subseteq \mathcal{B}$. Then*

$$\mathcal{A} \prec \mathcal{B} \iff \forall \phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}\, \forall a_1, \ldots, a_{n-1} \in A : (\mathcal{B} \models \phi(a_1, \ldots, a_{n-1}, b) \text{ for some } b \in B \\ \Rightarrow \mathcal{A} \models \phi(a_1, \ldots, a_{n-1}, a) \text{ for some } a \in A)$$

Theorem 3.5. *Let $\mathcal{L}$ be a language, κ, ν, λ be infinite cardinals with $\kappa \geqslant \nu \geqslant \max\{\lambda, \mathrm{card}\,(\mathcal{L})\}$, $\mathcal{B}$ be an $\mathcal{L}$-structure of cardinality κ and C be a subset of B of cardinality λ. Then there exists an $\mathcal{L}$-structure $\mathcal{A}$ of cardinality ν such that $\mathcal{B}|_C \subseteq \mathcal{A} \prec \mathcal{B}$.*

Proof. Using the Well-ordering Principle which is equivalent to the Axiom of Choice, let $<$ be a well-ordering on B. Define the sets

$$\begin{aligned}A_0 &:= \text{any subset of } B \text{ which contains } C \text{ and is of cardinality } \nu, \\ A_{n+1} &:= \{b \in B : \exists \phi(x_1, \ldots, x_{n+1}) \in \mathrm{Fml}_{\mathcal{L}}\, \exists a_1, \ldots, a_n \in A_n : b \text{ is the } <\text{-least} \\ &\qquad \text{element of } B \text{ such that } \mathcal{B} \models \phi(a_1, \ldots, a_n, b)\}, \text{ for } n \in \mathbb{N}.\end{aligned}$$

Take $a_1 \in A_n$. Observe that a_1 is the $<$-least element b of B such that $\mathcal{B} \models \phi(a_1, b)$, where ϕ is $x_1 = x_2$, since it is in fact the only such element. Therefore, we have $b \in B_{n+1}$ and more generally $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$.

Similarly, take an k-ary function symbol $f \in \mathcal{L}$ and $a_1, \ldots, a_k \in B_n$. Observe that $f^{\mathcal{B}}(a_1, \ldots, a_k)$ is the $<$-least element b of B such that $\mathcal{B} \models \phi(a_1, \ldots, a_k, b)$, where ϕ is $f(x_1, \ldots, x_k) = x_{k+1}$, since it is in fact the only such element. Therefore, we have $f^{\mathcal{M}}(a_1, \ldots, a_k) \in A_{n+1}$.

Hence, if we let

$$A := \bigcup_{n \in \mathbb{N}} A_n \quad \text{and} \quad \mathcal{A} := \mathcal{B}|_A,$$

we have that A is closed under the interpretations of function symbols and thus that the universe of $\mathcal{A}$ is indeed A.

Since $\nu \geqslant \mathrm{card}\,(\mathcal{L})$, we have by a quick induction on $n \in \mathbb{N}$ that

$$\mathrm{card}\,(A_{n+1}) = \mathrm{card}\,(\mathcal{L}) \times \underbrace{\mathrm{card}\,(A_n) \times \ldots \times \mathrm{card}\,(A_n)}_{n \text{ times}} = \nu.$$

Therefore, $\operatorname{card}(A) = \nu$.

We want to prove that $\mathcal{A}$ is the $\mathcal{L}$-structure that we are looking for. By construction, we already have $\mathcal{B}_{|_C} \subseteq \mathcal{A} \subseteq \mathcal{B}$. To further have that $\mathcal{A} \prec \mathcal{B}$, we will use the criterion given in Corollary 3.4. Take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$, $a_1, \ldots, a_{n-1} \in A$ and suppose that $\mathcal{B} \vDash \phi(a_1, \ldots, a_{n-1}, b)$ for some $b \in B$. For each $1 \leqslant i \leqslant n-1$, there exists $m_i \in \mathbb{N}$ such that $a_i \in A_{m_i}$. Let $m \coloneqq \max\{n, m_1, \ldots, m_{n-1}\}$, hence $a_1, \ldots, a_{n-1} \in A_m$. Observe that

$$b \in \{c \in B : \mathcal{B} \vDash \phi(a_1, \ldots, a_{n-1}, c)\}.$$

This set must contain an $<$-least element, say $a \in B$, which is therefore in A_n and thus in A. Hence $\mathcal{B} \vDash \phi(a_1, \ldots, a_{n-1}, a)$ and by Corollary 3.4, this concludes the proof. □

Definition 3.6. For a set A, and an ordinal β, we say that $a \in {}^{\beta}A$ is an *enumeration* of A if a is surjective. And we say that an enumeration a is an *enumeration without repetitions* if it is injective.

Notation. Let β be an ordinal, $\mathcal{L}$ be a language, $\mathcal{A}$ be an $\mathcal{L}$-structure and $a \in {}^{\beta}A$. We denote a new language

$$\mathcal{L}_\beta \coloneqq \mathcal{L} \cup \{c_\alpha : \alpha < \beta\},$$

where c_α is a new constant symbol for each $\alpha < \beta$. Observe that a gives a possible interpretation of these new constant symbols. This motivates the notation $(\mathcal{A}, a)$ for the $\mathcal{L}_\beta$-structure $\mathcal{A}'$ defined as follows:

- Its universe is $A' \coloneqq A$.
- For each constant symbol $c \in \mathcal{L}$, function symbol $f \in \mathcal{L}$ and predicate symbol $R \in \mathcal{L}$, we have:
$$c^{\mathcal{A}'} \coloneqq c^{\mathcal{A}}, \quad f^{\mathcal{A}'} \coloneqq f^{\mathcal{A}}, \quad R^{\mathcal{A}'} \coloneqq R^{\mathcal{A}}.$$
- For each $\alpha < \beta$, $c_\alpha^{\mathcal{A}'} \coloneqq a(\alpha)$.

Similarly, whenever we have a new n-ary predicate symbol P and a set $X \subseteq A^n$, we will use the notation $(\mathcal{A}, X)$ for the $(\mathcal{L} \cup \{P\})$-structure which is in all point similar to $\mathcal{A}$ and has in addition $P^{(\mathcal{A},X)} \coloneqq X$.

Lemma 3.7 (Frayne's Lemma)**.** *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures. Then*

$$\mathcal{A} \equiv \mathcal{B} \iff \exists \text{non-empty set } I, \exists \text{ultrafilter } \mathcal{U} \text{ such that } \mathcal{B} \text{ is elementarily embeddable in } {}^{I}\mathcal{A}/_{\mathcal{U}}.$$

Proof. Let us denote the ultraproduct as such:

$$\mathcal{A}' \coloneqq {}^{I}\mathcal{A}/_{\mathcal{U}}.$$

($\Leftarrow$) Let I be a non-empty set, $\mathcal{U}$ be an ultrafilter on I, and suppose that $\mathcal{B}$ is elementarily embeddable in $\mathcal{A}'$. Then

$$\mathcal{B} \equiv \mathcal{A}' \overset{\text{Łoš's Theorem}}{\equiv} \mathcal{A},$$

as desired.

($\Rightarrow$) Suppose that $\mathcal{A} \equiv \mathcal{B}$. Let $b \in {}^{\alpha}B$ be an enumeration of B, and define

$$I := \{\sigma \in \mathrm{Sent}_{\mathcal{L}_\alpha} : (\mathcal{B}, b) \vDash \sigma)\} = \mathrm{Th}_{\mathcal{L}_\alpha}(\mathcal{B}, b).$$

Take $\sigma \in I$. It contains finitely many constant symbols from $\mathcal{L}_\alpha \setminus \mathcal{L}$, say $c_{\beta_1}, \ldots, c_{\beta_n}$ for some $\beta_1, \ldots, \beta_n < \alpha$. Replace them respectively by new variables $y_1, \ldots, y_n$ to obtain an $\mathcal{L}$-formula $\phi(y_1, \ldots, y_n)$. Let ρ be the $\mathcal{L}$-sentence

$$\exists y_1 \ldots \exists y_n \phi.$$

Then,

$$\begin{aligned}(\mathcal{B}, b) \vDash \sigma &\Longrightarrow \mathcal{B} \vDash \rho \\ &\overset{\mathcal{A}\equiv\mathcal{B}}{\Longrightarrow} \mathcal{A} \vDash \rho \\ &\Longrightarrow \text{there exists } a_\sigma \in {}^{\alpha}A \text{ such that } (\mathcal{A}, a_\sigma) \vDash \sigma.\end{aligned}$$

Since $\sigma \in I$ is arbitrary, we can define

$$J_\sigma := \{\tau \in I : (\mathcal{A}, a_\tau) \vDash \sigma\},$$

which is non-empty since $a_\sigma \in J_\sigma$. We claim that the collection $J := \{J_\sigma : \sigma \in I\}$ has the fip. Indeed, take $\sigma_1, \ldots, \sigma_n \in I$, then

$$\begin{aligned}J_{\sigma_1} \cap \ldots \cap J_{\sigma_n} &= \{\tau \in I : (\mathcal{A}, a_\tau) \vDash \sigma_i \text{ for } 1 \leqslant i \leqslant n\} \\ &= \{\tau \in I : (\mathcal{A}, a_\tau) \vDash \sigma_1 \wedge \ldots \wedge \sigma_n\} \\ &= J_{\sigma_1 \wedge \ldots \wedge \sigma_n} \neq \varnothing.\end{aligned}$$

Therefore, by Proposition 2.20 and Theorem 2.23, J is contained in an ultrafilter $\mathcal{U}$ on I. Now that we have a well-defined ultraproduct $\mathcal{A}'$, we can define a map

$$H : \mathcal{B} \to \mathcal{A}' : x \mapsto [f_\beta : I \to A : \sigma \mapsto a_\sigma(\beta)],$$

where $\beta < \alpha$ is such that $x = b_\beta$. Let us check that H is an elementary embedding.

Well-defined: Suppose that $b_\beta = b_\gamma$ for some $\beta < \gamma < \alpha$. It means that $(\mathcal{B}, b) \vDash c_\beta = c_\gamma$, i.e., $(c_\beta = c_\gamma) \in I$. Observe that

$$\begin{aligned}\{\sigma \in I : f_\beta(\sigma) = f_\gamma(\sigma)\} &= \{\sigma \in I : a_\sigma(\beta) = a_\sigma(\gamma)\} \\ &= \{\sigma \in I : (\mathcal{A}, a_\sigma) \vDash c_\beta = c_\gamma\} \\ &= J_{c_\beta = c_\gamma} \in \mathcal{U},\end{aligned}$$

which implies that $f_\beta \sim f_\gamma$, or in other words that

$$H(b_\beta) = [f_\beta] = [f_\gamma] = H(b_\gamma).$$

Injectivity: Similarly, take $\beta < \gamma < \alpha$ and suppose that b_β, b_γ are distinct. Then $(\mathcal{B}, b) \vDash \neg c_\beta = c_\gamma$, i.e., $(\neg c_\beta = c_\gamma) \in I$, and thus

$$\begin{aligned} \{\sigma \in I : f_\beta(\sigma) \neq f_\gamma(\sigma)\} &= \{\sigma \in I : a_\sigma(\beta) \neq a_\sigma(\gamma)\} \\ &= \{\sigma \in I : (\mathcal{A}, a_\sigma) \vDash \neg c_\beta = c_\gamma\} \\ &= J_{\neg c_\beta = c_\gamma} \in \mathcal{U}. \end{aligned}$$

By maximality of $\mathcal{U}$, we have $\{\sigma \in I : f_\beta(\sigma) = f_\gamma(\sigma)\} \notin \mathcal{U}$, which proves that $f_\beta \nsim f_\gamma$ and that $H(b_\beta) \neq H(b_\gamma)$.

Embedding: Let us check the three conditions.

- Let $c \in \mathcal{L}$ be a constant symbol. We have $c^{\mathcal{B}} = b_\beta$ for some $\beta < \alpha$. It means that $(\mathcal{B}, b) \vDash c_\beta = c$, i.e., $(c_\beta = c) \in I$. Observe that

  $$H(c^{\mathcal{B}}) = H(b_\beta) = [f_\beta] = \left[g_\beta : I \to A : \sigma \mapsto c^{\mathcal{A}}\right] \overset{\text{def. ultraproduct}}{=} c^{\mathcal{A}'},$$

  with the last but one equality holding because

  $$\begin{aligned} \{\sigma \in I : f_\beta(\sigma) = g_\beta(\sigma)\} &= \left\{\sigma \in I : a_\sigma(\beta) = c^{\mathcal{A}}\right\} \\ &= \{\sigma \in I : (\mathcal{A}, a_\sigma) \vDash c_\beta = c\} \\ &= J_{c_\beta = c} \in \mathcal{U}. \end{aligned}$$

- Let $h \in \mathcal{L}$ be an n-ary function symbol and $b_{\beta_1}, \ldots, b_{\beta_n} \in B$, where $\beta_1, \ldots, \beta_n < \alpha$. Observe that for some $\beta < \alpha$, we have

  $$\begin{aligned} h^{\mathcal{B}}(b_{\beta_1}, \ldots, b_{\beta_n}) = b_\beta &\Longrightarrow (\mathcal{B}, b) \vDash h(c_{\beta_1}, \ldots, c_{\beta_n}) = c_\beta \\ &\Longrightarrow (h(c_{\beta_1}, \ldots, c_{\beta_n}) = c_\beta) \in I. \end{aligned}$$

  Moreover,

  $$\begin{aligned} \left\{\sigma \in I : h^{\mathcal{A}}(f_{\beta_1}(\sigma), \ldots, f_{\beta_n}(\sigma)) = f_\beta(\sigma)\right\} &= \left\{\sigma \in I : h^{\mathcal{A}}(a_\sigma(\beta_1), \ldots, a_\sigma(\beta_n)) = a_\sigma(\beta)\right\} \\ &= \{\sigma \in I : (\mathcal{A}, a_\sigma) \vDash h(c_{\beta_1}, \ldots, c_{\beta_n}) = c_\beta\} \\ &= J_{h(c_{\beta_1}, \ldots, c_{\beta_n}) = c_\beta} \in \mathcal{U}. \end{aligned}$$

  Therefore,

  $$\begin{aligned} H(h^{\mathcal{B}}(b_{\beta_1}, \ldots, b_{\beta_n})) &= H(b_\beta) = [f_\beta] \\ &= \left[I \to A : \sigma \mapsto h^{\mathcal{A}}(f_{\beta_1}(\sigma), \ldots, f_{\beta_n}(\sigma))\right] \\ &\overset{\text{def. ultraproduct}}{=} h^{\mathcal{A}'}([f_{\beta_1}], \ldots, [f_{\beta_n}]) \\ &= h^{\mathcal{A}'}(H(b_{\beta_1}), \ldots, H(b_{\beta_n})). \end{aligned}$$

- Let $R \in \mathcal{L}$ be an n-ary predicate symbol and $b_{\beta_1}, \ldots, b_{\beta_n} \in B$, where $\beta_1, \ldots, \beta_n < \alpha$. Observe that

$$
\begin{aligned}
(H(b_{\beta_1}), \ldots, H(b_{\beta_n})) \in R^{\mathcal{A}'} &\iff ([f_{\beta_1}], \ldots, [f_{\beta_n}]) \in R^{\mathcal{A}'} \\
&\overset{\text{def. ultraproduct}}{\iff} \left\{\sigma \in I : (f_{\beta_1}(\sigma), \ldots, f_{\beta_n}(\sigma)) \in R^{\mathcal{A}}\right\} \in \mathcal{U} \\
&\iff \left\{\sigma \in I : (a_\sigma(\beta_1), \ldots, a_\sigma(\beta_n)) \in R^{\mathcal{A}}\right\} \in \mathcal{U} \\
&\iff \{\sigma \in I : (\mathcal{A}, a_\sigma) \vDash R(c_{\beta_1}, \ldots, c_{\beta_n})\} \in \mathcal{U} \\
&\iff J_{R(c_{\beta_1}, \ldots, c_{\beta_n})} \in \mathcal{U} \\
&\iff R(c_{\beta_1}, \ldots, c_{\beta_n}) \in I \\
&\iff (\mathcal{B}, b) \vDash R(c_{\beta_1}, \ldots, c_{\beta_n}) \\
&\iff (b_{\beta_1}, \ldots, b_{\beta_n}) \in R^{\mathcal{B}}
\end{aligned}
$$

Elementary embedding: Take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $b_{\beta_1}, \ldots, b_{\beta_n} \in B$, where $\beta_1, \ldots, \beta_n < \alpha$. Observe that

$$
\begin{aligned}
\mathcal{A}' \vDash \phi(H(b_{\beta_1}), \ldots, H(b_{\beta_n})) &\iff \mathcal{A}' \vDash \phi([f_{\beta_1}], \ldots, [f_{\beta_n}]) \\
&\overset{\text{Łoš's Theorem}}{\iff} \{\sigma \in I : \mathcal{A} \vDash \phi(f_{\beta_1}(\sigma), \ldots, f_{\beta_n}(\sigma))\} \in \mathcal{U} \\
&\iff \{\sigma \in I : \mathcal{A} \vDash \phi(a_\sigma(\beta_1), \ldots, a_\sigma(\beta_n))\} \in \mathcal{U} \\
&\iff \left\{\sigma \in I : (\mathcal{A}, a_\sigma) \vDash \phi\left[\tfrac{c_{\beta_1}}{x_1} \ldots \tfrac{c_{\beta_n}}{x_n}\right]\right\} \in \mathcal{U} \\
&\iff J_{\phi\left[\frac{c_{\beta_1}}{x_1} \ldots \frac{c_{\beta_n}}{x_n}\right]} \in \mathcal{U} \\
&\iff \phi\left[\tfrac{c_{\beta_1}}{x_1} \ldots \tfrac{c_{\beta_n}}{x_n}\right] \in I \\
&\iff (\mathcal{B}, b) \vDash \phi\left[\tfrac{c_{\beta_1}}{x_1} \ldots \tfrac{c_{\beta_n}}{x_n}\right] \\
&\iff \mathcal{B} \vDash \phi(b_{\beta_1}, \ldots, b_{\beta_n}),
\end{aligned}
$$

which proves that H is an elementary embedding and concludes the proof. □

Corollary 3.8. *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be finite $\mathcal{L}$-structures. Then*

$$\mathcal{A} \equiv \mathcal{B} \iff \mathcal{A} \cong \mathcal{B}.$$

Proof. ($\Leftarrow$) This direction is a direct consequence of Proposition 3.2.

($\Rightarrow$) Suppose that $\mathcal{A} \equiv \mathcal{B}$. By Frayne's Lemma, there exists a non-empty set I and an ultrafilter $\mathcal{U}$ on I such that $\mathcal{B}$ is elementary embeddable in ${}^{I}\mathcal{A}/\mathcal{U}$. Since $\mathcal{A}$ is finite, by Lemma 2.28 we have

$$\operatorname{card}(\mathcal{A}) = \operatorname{card}\left({}^{I}\mathcal{A}/\mathcal{U}\right).$$

Since $\mathcal{A} \equiv \mathcal{B}$, we also have $\operatorname{card}(\mathcal{A}) = \operatorname{card}(\mathcal{B})$, because of the sentence requiring to have exactly the right finite number of elements. Therefore, the elementary embedding $\mathcal{B} \to {}^{I}\mathcal{A}/\mathcal{U}$ must be surjective, i.e., is an isomorphism, which concludes the proof. □

Lemma 3.9. *Let $\mathcal{L}$ be a language, $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures, α be an ordinal and $a \in {}^{\alpha}A$. If $\mathcal{A} \prec \mathcal{B}$, then $(\mathcal{A}, a) \prec (\mathcal{B}, a)$.*

Proof. Observe first that since $A \subseteq B$, then a can also be viewed as an element of ${}^{\alpha}B$. Take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}_\alpha}$ and $a_1, \ldots, a_n \in A$. There is only finitely many constants from $\mathcal{L}_\alpha \setminus \mathcal{L}$ occuring in ϕ, say $c_{\alpha_1}, \ldots, c_{\alpha_m}$ for some $\alpha_1, \ldots, \alpha_m < \alpha$. Replace them respectively by new variables $y_1, \ldots, y_m$ to obtain an $\mathcal{L}$-formula $\psi(x_1, \ldots, x_n, y_1, \ldots, y_m)$. Then,

$$\begin{aligned}(\mathcal{A}, a) \vDash \phi(a_1, \ldots, a_n) &\iff \mathcal{A} \vDash \psi(a_1, \ldots, a_n, a(\alpha_1), \ldots, a(\alpha_m)) \\ &\overset{\mathcal{A} \prec \mathcal{B}}{\iff} \mathcal{B} \vDash \psi(a_1, \ldots, a_n, a(\alpha_1), \ldots, a(\alpha_m)) \\ &\iff (\mathcal{B}, a) \vDash \phi(a_1, \ldots, a_n).\end{aligned}$$

In other words, $(\mathcal{A}, a) \prec (\mathcal{B}, a)$, which concludes the proof. □

Lemma 3.10. *Let $\mathcal{L}$ be a language, $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures and $a \in {}^{\beta}A$ be an enumeration of A. Then:*

$$\mathcal{A} \textit{ is elementarily embedabble in } \mathcal{B} \iff \textit{there exists } b \in {}^{\beta}B \textit{ such that} : (\mathcal{A}, a) \equiv_{\mathcal{L}_\beta} (\mathcal{B}, b).$$

Proof. ($\Longrightarrow$) Suppose there is an elementary embedding $H : \mathcal{A} \to \mathcal{B}$. For each $\alpha < \beta$, let $b(\alpha) \coloneqq H(a(\alpha))$. This defines a $b \in {}^{\beta}B$. Let us check that $(\mathcal{A}, a) \equiv_{\mathcal{L}_\beta} (\mathcal{B}, b)$. Take $\sigma \in \mathrm{Sent}_{\mathcal{L}_\beta}$, which must be of the form $\phi\left[\frac{c_{\alpha_1}}{x_1} \ldots \frac{c_{\alpha_n}}{x_n}\right]$ for some $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and some constants $c_{\alpha_1}, \ldots, c_{\alpha_n} \in \mathcal{L}_\beta$. Therefore,

$$\begin{aligned}(\mathcal{A}, a) \vDash \sigma &\iff (\mathcal{A}, a) \vDash \phi\left[\tfrac{c_{\alpha_1}}{x_1} \ldots \tfrac{c_{\alpha_n}}{x_n}\right] \\ &\iff \mathcal{A} \vDash \phi(a(\alpha_1), \ldots, a(\alpha_n)) \\ &\overset{H \text{ elem. emb.}}{\iff} \mathcal{B} \vDash \phi(b(\alpha_1), \ldots, b(\alpha_n)) \\ &\iff (\mathcal{B}, b) \vDash \phi\left[\tfrac{c_{\alpha_1}}{x_1} \ldots \tfrac{c_{\alpha_n}}{x_n}\right] \\ &\iff (\mathcal{B}, b) \vDash \sigma.\end{aligned}$$

($\Leftarrow$) Suppose now that there exists $b \in {}^{\beta}B$ such that $(\mathcal{A}, a) \equiv_{\mathcal{L}_\beta} (\mathcal{B}, b)$. Define $H : A \to B : x \mapsto b_\alpha$, where $\alpha < \beta$ is such that $x = a_\alpha$.

Well-defined and injectivity: Let $\alpha < \alpha' < \beta$. Observe that

$$\begin{aligned}a_\alpha = a_{\alpha'} &\iff (\mathcal{A}, a) \vDash c_\alpha = c_{\alpha'} \\ &\overset{\mathcal{L}_\beta\text{-elem. equiv.}}{\iff} (\mathcal{B}, b) \vDash c_\alpha = c_{\alpha'} \\ &\iff b_\alpha = b_{\alpha'},\end{aligned}$$

which proves that H is well-defined and injective.

Embedding: Let us check the three conditions.

- Let $c \in \mathcal{L}$ be a constant symbol. We have that $c^{\mathcal{A}} = a_\alpha$ for some $\alpha < \beta$. Observe that

$$\begin{aligned} c^{\mathcal{A}} = a_\alpha &\Longrightarrow (\mathcal{A}, a) \vDash c = c_\alpha \\ &\overset{\mathcal{L}_\beta\text{-elem. equiv.}}{\Longrightarrow} (\mathcal{B}, b) \vDash c = c_\alpha \\ &\Longrightarrow c^{\mathcal{B}} = b_\alpha = H(a_\alpha) = H(c^{\mathcal{A}}). \end{aligned}$$

- Let $f \in \mathcal{L}$ be an n-ary function symbol and $a_{\alpha_1}, \dots, a_{\alpha_n} \in A$, where $\alpha_1, \dots, \alpha_n < \beta$. We have that $f^{\mathcal{A}}(a_{\alpha_1}, \dots, a_{\alpha_n}) = a_\alpha$ for some $\alpha < \beta$. Observe that

$$\begin{aligned} f^{\mathcal{A}}(a_{\alpha_1}, \dots, a_{\alpha_n}) = a_\alpha &\Longrightarrow (\mathcal{A}, a) \vDash f(c_{\alpha_1}, \dots, c_{\alpha_n}) = c_\alpha \\ &\overset{\mathcal{L}_\beta\text{-elem. equiv.}}{\Longrightarrow} (\mathcal{B}, b) \vDash f(c_{\alpha_1}, \dots, c_{\alpha_n}) = c_\alpha \\ &\Longrightarrow f^{\mathcal{B}}(H(a_{\alpha_1}), \dots, H(a_{\alpha_n})) = f^{\mathcal{B}}(b_{\alpha_1}, \dots, b_{\alpha_n}) = b_\alpha = H(a_\alpha) \\ &\qquad = H(f^{\mathcal{A}}(a_{\alpha_1}, \dots, a_{\alpha_n})) \end{aligned}$$

- Let $R \in \mathcal{L}$ be an n-ary predicate symbol and $a_{\alpha_1}, \dots, a_{\alpha_n} \in A$, where $\alpha_1, \dots, \alpha_n < \beta$. Observe that

$$\begin{aligned} (a_{\alpha_1}, \dots, a_{\alpha_n}) \in R^{\mathcal{A}} &\Longleftrightarrow (\mathcal{A}, a) \vDash R(c_{\alpha_1}, \dots, c_{\alpha_n}) \\ &\overset{\mathcal{L}_\beta\text{-elem. equiv.}}{\Longleftrightarrow} (\mathcal{B}, b) \vDash R(c_{\alpha_1}, \dots, c_{\alpha_n}) \\ &\Longleftrightarrow (b_{\alpha_1}, \dots, b_{\alpha_n}) \in R^{\mathcal{B}} \\ &\Longleftrightarrow (H(a_{\alpha_1}), \dots, H(a_{\alpha_n})) \in R^{\mathcal{B}}. \end{aligned}$$

Elementary embedding: Take $\phi(x_1, \dots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a_{\alpha_1}, \dots, a_{\alpha_n} \in A$, where $\alpha_1, \dots, \alpha_n < \beta$. Observe that

$$\begin{aligned} \mathcal{A} \vDash \phi(a_{\alpha_1}, \dots, a_{\alpha_n}) &\Longleftrightarrow (\mathcal{A}, a) \vDash \phi(c_{\alpha_1}, \dots, c_{\alpha_n}) \\ &\overset{\mathcal{L}_\beta\text{-elem. equiv.}}{\Longleftrightarrow} (\mathcal{B}, b) \vDash \phi(c_{\alpha_1}, \dots, c_{\alpha_n}) \\ &\Longleftrightarrow \mathcal{B} \vDash \phi(b_{\alpha_1}, \dots, b_{\alpha_n}) \\ &\Longleftrightarrow \mathcal{B} \vDash \phi(H(a_{\alpha_1}), \dots, H(a_{\alpha_n})). \end{aligned}$$

We have proved everything to know that H is indeed an elementary embedding, which finishes the proof. □

Theorem 3.11. *Let $\mathcal{L}$ be a language, $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures of same cardinality, and $a \in {}^{\beta}A$ and $b \in {}^{\beta}B$ be enumerations. If $(\mathcal{A}, a) \equiv_{\mathcal{L}_\beta} (\mathcal{B}, b)$, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. By Lemma 3.10, we know the existence of the following well-defined elementary embedding $H : A \to B : x \mapsto b_\alpha$, where $\alpha < \beta$ is such that $x = a_\alpha$. Since here a, b are enumerations, H is in particular a surjective embedding, i.e., an isomorphism. □

Definition 3.12. Let $\mathcal{L}$ be a language. A sequence $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle$ of $\mathcal{L}$-structures is a *chain* if

$$\forall \alpha \leqslant \beta < \gamma : \mathcal{A}_\alpha \subseteq \mathcal{A}_\beta.$$

The *union of the chain* $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle$ is the $\mathcal{L}$-structure denoted

$$\mathcal{A} \coloneqq \bigcup_{\alpha<\gamma} \mathcal{A}_\alpha,$$

and defined as follows.

- The universe is $A = \bigcup\limits_{\alpha<\gamma} A_\alpha$
- For each constant symbol $c \in \mathcal{L}$, $c^{\mathcal{A}} = c^{\mathcal{A}_0}$, so in particular $c^{\mathcal{A}} = c^{\mathcal{A}_\alpha}$ for all $\alpha < \gamma$.
- For each function symbol $f \in \mathcal{L}$, $f^{\mathcal{A}} \coloneqq \bigcup\limits_{\alpha<\gamma} f^{\mathcal{A}_\alpha}$, which is well-defined since for all $\alpha < \beta < \gamma$, $f^{\mathcal{A}_\beta}$ extends $f^{\mathcal{A}_\alpha}$.
- For each predicate symbol $R \in \mathcal{L}$, let $R^{\mathcal{A}} \coloneqq \bigcup\limits_{\alpha<\gamma} R^{\mathcal{A}_\alpha}$.

The chain is an *elementary chain* if

$$\forall \alpha \leqslant \beta < \gamma : \mathcal{A}_\alpha \prec \mathcal{A}_\beta.$$

Theorem 3.13 (Theorem of union of chains)**.** *The union of an elementary chain is an elementary extension of each element of the chain.*

Proof. Take an elementary chain $\langle \mathcal{A}_\alpha : \alpha < \gamma \rangle$. Denote by $\mathcal{A}$ the union of this chain. Observe the following:

- For each $\alpha < \gamma$:
$$A_\alpha \subseteq \bigcup_{\beta<\gamma} A_\beta = A$$
- For each constant symbol $c \in \mathcal{L}$ and each $\alpha < \gamma$, $c^{\mathcal{A}} = c^{\mathcal{A}_\alpha}$ by definition.
- For each n-ary function symbol $f \in \mathcal{L}$ and each $\alpha < \gamma$:
$$f^{\mathcal{A}}|_{A_\alpha^n} = \bigcup_{\beta<\gamma} f^{\mathcal{A}_\beta}|_{A_\alpha^n} = \bigcup_{\beta\leqslant\alpha} f^{\mathcal{A}_\beta} = f^{\mathcal{A}_\alpha}.$$

- For each n-ary predicate symbol $R \in \mathcal{L}$ and each $\alpha < \gamma$:

$$\begin{aligned} A_\alpha^n \cap R^{\mathcal{A}} &= A_\alpha^n \cap \left(\bigcup_{\beta<\gamma} R^{\mathcal{A}_\beta} \right) = \bigcup_{\beta<\gamma} \left(A_\alpha^n \cap R^{\mathcal{A}_\beta} \right) \\ &= \underbrace{\bigcup_{\beta<\alpha} \left(A_\alpha^n \cap R^{\mathcal{A}_\beta} \right)}_{\subseteq R^{\mathcal{A}_\alpha}} \cup \underbrace{\bigcup_{\alpha \leqslant \beta<\gamma} \left(A_\alpha^n \cap R^{\mathcal{A}_\beta} \right)}_{= R^{\mathcal{A}_\alpha}} \\ &= R^{\mathcal{A}_\alpha} \end{aligned}$$

So $\mathcal{A}$ is indeed an extension of each element of the chain. Now, we will see that those extensions are in fact elementary extensions. Fix $\alpha < \gamma$. First, let $t(x_1, \ldots, x_n) \in \mathrm{Tm}_{\mathcal{L}}$ and $a_1, \ldots, a_n \in A$. We will prove the following claim by induction on the complexity of t:

Claim: $t^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) = t^{\mathcal{A}}(a_1, \ldots, a_n)$.

- If t is x_i for some $1 \leqslant i \leqslant n$, then

$$t^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) = a_i = t^{\mathcal{A}}(a_1, \ldots, a_n).$$

- If t is c where $c \in \mathcal{L}$ is a constant symbol, then

$$t^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) = c^{\mathcal{A}_\alpha} = c^{\mathcal{A}} = t^{\mathcal{A}}(a_1, \ldots, a_n).$$

- If t is $f(t_1, \ldots, t_k)$ where $f \in \mathcal{L}$ is a k-ary function symbol, then

$$\begin{aligned} t^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) &= f^{\mathcal{A}_\alpha}\left(t_1^{\mathcal{A}_\alpha}(a_1, \ldots, a_n), \ldots, t_k^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) \right) \\ &\overset{\text{I.H. and def.}}{=} f^{\mathcal{A}}\left(t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_k^{\mathcal{A}}(a_1, \ldots, a_n) \right) \\ &= t^{\mathcal{A}}(a_1, \ldots, a_n). \end{aligned}$$

Now, take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a_1, \ldots, a_n \in A_\alpha$. Let us prove by induction on the complexity of ϕ that

$$\mathcal{A}_\alpha \vDash \phi(a_1, \ldots, a_n) \iff \mathcal{A} \vDash \phi(a_1, \ldots, a_n).$$

- If ϕ is $t_1 = t_2$:

$$\begin{aligned} \mathcal{A}_\alpha \vDash \phi(a_1, \ldots, a_n) &\iff t_1^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) = t_2^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) \\ &\overset{\text{Claim}}{\iff} t_1^{\mathcal{A}}(a_1, \ldots, a_n) = t_2^{\mathcal{A}}(a_1, \ldots, a_n) \\ &\iff \mathcal{A} \vDash \phi(a_1, \ldots, a_n). \end{aligned}$$

- If ϕ is $R(t_1, \ldots, t_m)$:

$$\begin{aligned} \mathcal{A}_\alpha \vDash \phi(a_1, \ldots, a_n) &\iff \left(t_1^{\mathcal{A}_\alpha}(a_1, \ldots, a_n), \ldots, t_m^{\mathcal{A}_\alpha}(a_1, \ldots, a_n) \right) \in R^{\mathcal{A}_\alpha} = A_\alpha^m \cap R^{\mathcal{A}} \\ &\overset{\text{Claim}}{\iff} \left(t_1^{\mathcal{A}}(a_1, \ldots, a_n), \ldots, t_m^{\mathcal{A}}(a_1, \ldots, a_n) \right) \in R^{\mathcal{A}} \\ &\iff \mathcal{A} \vDash \phi(a_1, \ldots, a_n). \end{aligned}$$

- If ϕ is $\neg\psi$:

$$\begin{aligned}\mathcal{A}_\alpha \vDash \phi(a_1,\dots,a_n) &\iff \mathcal{A}_\alpha \nvDash \psi(a_1,\dots,a_n)\\ &\overset{\text{I.H.}}{\iff} \mathcal{A} \nvDash \psi(a_1,\dots,a_n)\\ &\iff \mathcal{A} \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

- If ϕ is $\psi \wedge \theta$:

$$\begin{aligned}\mathcal{A}_\alpha \vDash \phi(a_1,\dots,a_n) &\iff \mathcal{A}_\alpha \vDash \psi(a_1,\dots,a_n) \text{ and } \mathcal{A}_\alpha \vDash \theta(a_1,\dots,a_n)\\ &\overset{\text{I.H.}}{\iff} \mathcal{A} \vDash \psi(a_1,\dots,a_n) \text{ and } \mathcal{A} \vDash \theta(a_1,\dots,a_n)\\ &\iff \mathcal{A} \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

- If ϕ is $\exists x \psi(x, x_1, \dots, x_n)$:

$$\begin{aligned}\mathcal{A}_\alpha \vDash \phi(a_1,\dots,a_n) &\implies \text{for some } a \in A_\alpha : \mathcal{A}_\alpha \vDash \psi(a, a_1,\dots,a_n)\\ &\overset{\text{I.H.}}{\implies} \text{for some } a \in A_\alpha \subseteq A : \mathcal{A} \vDash \psi(a, a_1,\dots,a_n)\\ &\implies \mathcal{A} \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

  Conversely, suppose that $\mathcal{A} \vDash \phi(a_1,\dots,a_n)$. Then, there exists $a \in A = \bigcup_{\beta<\gamma} A_\beta$ such that $\mathcal{A} \vDash \psi(a, a_1,\dots,a_n)$. Let $\beta < \gamma$ be such that $a \in A_\beta$, and $\beta' := \max\{\alpha, \beta\}$. Then $a \in A_{\beta'}$, and by I.H., $A_{\beta'} \vDash \psi(a, a_1,\dots,a_n)$. Hence $A_{\beta'} \vDash \phi(a_1,\dots,a_n)$. As $\alpha \leqslant \beta' < \gamma$ and $\mathcal{A}_\alpha \prec \mathcal{A}_{\beta'}$, we know that $\mathcal{A}_\alpha \vDash \phi(a_1,\dots,a_n)$, which concludes the proof.

□

3.2 Set theory prerequisites

Definition 3.14. Let $(A, <)$ be a poset. A subset $B \subseteq A$ is *cofinal* in A if

$$\forall a \in A, \exists b \in B : a \leqslant b.$$

The *cofinality* of A is defined as

$$cf(A) := \min\{\operatorname{card}(B) \mid B \subseteq A \text{ is cofinal in } A\}.$$

It is well-defined because of the well-ordering principle. Observe that we always have $cf(A) \leqslant \operatorname{card}(A)$. In the case of an ordinal α, it is said to be *regular* when $cf(\alpha) = \alpha$ and *singular* otherwise when $cf(\alpha) < \alpha$.

Proposition 3.15. *Every successor cardinal is regular.*

Proof. Let κ be a cardinal. Suppose ab absurdum that there exists $A \subseteq \kappa^+$ which is cofinal in κ^+ and with $\lambda := \operatorname{card}(A) < \kappa^+$. Let $a \in {}^{\lambda}A$ be an enumeration without repetition. Since $\lambda < \kappa^+$, we have $\lambda \leqslant \kappa$. Note that for $\alpha < \lambda$, $a(\alpha) \in A \subseteq \kappa^+$, so it has cardinality $\leqslant \kappa$ and thus $a(\alpha) \preccurlyeq \kappa$. Therefore,

$$\kappa^+ \overset{\text{Cofinality}}{=} \bigcup_{\alpha<\lambda} a(\alpha) \leqslant \lambda \cdot \kappa = \max\{\lambda, \kappa\} = \kappa \ \lightning.$$

□

Definition 3.16. The *beth numbers* are cardinal numbers defined recursively as follows.

- $\beth_0 := \aleph_0$
- If $\alpha = \beta + 1$ is a successor ordinal, $\beth_{\beta+1} := 2^{\beth_\beta}$.
- If α is a limit ordinal, $\beth_\alpha := \sup\{\beth_\beta : \beta < \alpha\}$

A *limit beth number* is a beth number $\beth_\alpha$ indexed by a limit ordinal α. Observe that because of Cantor's theorem, we have that for every ordinal α: $\beth_\alpha < 2^{\beth_\alpha} = \beth_{\alpha+1}$.

Definition 3.17. If κ and λ are cardinals, we define

$$\lambda^{<\kappa} := \sup\{\lambda^\mu : \mu \text{ is a cardinal and } \mu < \kappa\}.$$

Proposition 3.18. *Let κ be a limit Beth number. Then $2^{<\kappa} = \kappa$.*

Proof. We have $\kappa = \beth_\alpha$ for some limit ordinal α.

($\leqslant$) Take $\mu < \kappa = \beth_\alpha = \sup\{\beth_\beta : \beta < \alpha\}$. By definition of the supremum, there exists $\gamma < \alpha$ such that $\mu < \beth_\gamma$. Therefore,

$$2^\mu < 2^{\beth_\gamma} = \beth_{\gamma+1} \leqslant \sup\{\beth_\beta : \beta < \alpha\} = \beth_\alpha = \kappa.$$

By passing to the supremum, we obtain

$$2^{<\kappa} \leqslant \kappa.$$

($\geqslant$) Take $\beta < \alpha$. We have

$$\beth_\beta < \beth_{\beta+1} \leqslant \sup\{\beth_\gamma : \gamma < \alpha\} = \kappa,$$

and hence

$$\beth_{\beta+1} = 2^{\beth_\beta} \in \{2^\mu : \mu < \kappa\}.$$

Therefore,

$$\begin{aligned} \kappa = \beth_\alpha &= \sup\{\beth_\beta : \beta < \alpha\} \\ &= \sup\{\beth_{\beta+1} : \beta < \alpha\} \\ &\leqslant \sup\{2^\mu : \mu < \kappa\} = 2^{<\kappa}. \end{aligned}$$

□

3.3 Homogeneous structures

Definition 3.19. Let $\mathcal{L}$ be a language and κ be a cardinal. An $\mathcal{L}$-structure $\mathcal{A}$ is *κ-homogeneous* if for all ordinal $\beta < \kappa$, all $B = \{b_\alpha : \alpha < \beta\} \subseteq A$ with $\text{card}\,(B) < \kappa$ and all map $f : B \to A$,

$$(\mathcal{A}, b) \equiv (\mathcal{A}, f(b)) \Longrightarrow f \text{ can be extended to an automorphism of } \mathcal{A},$$

where $b : \beta \to A : \alpha \mapsto b_\alpha$ and $f(b) : \beta \to A : \alpha \mapsto f(b_\alpha)$. We say that $\mathcal{A}$ is *homogeneous* if it is κ-homogeneous where $\kappa = \text{card}\,(A)$.

Lemma 3.20. *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be isomorphic $\mathcal{L}$-structures. If $\mathcal{A}$ is homogeneous, then so is $\mathcal{B}$.*

Proof. Let $H : \mathcal{B} \cong \mathcal{A}$ be an isomorphism. Take $\beta < \text{card}\,(\mathcal{B})$, $C = \{b_\alpha : \alpha < \beta\} \subseteq B$ and $f : C \to B$ such that $(\mathcal{B}, b) \equiv (\mathcal{B}, f(b))$, where $b : \beta \to B : \alpha \mapsto b_\alpha$ and $f(b) : \beta \to B : \alpha \mapsto f(b_\alpha)$. Let also $H(b) : \beta \to B : \alpha \mapsto H(b_\alpha)$ and $H(f(b)) : \beta \to B : \alpha \mapsto H(f(b_\alpha))$. Then,

$$H : (\mathcal{B}, b) \cong (\mathcal{A}, H(b))$$
$$H : (\mathcal{B}, f(b)) \cong (\mathcal{A}, H(f(b))).$$

Hence,

$$(\mathcal{A}, H(b)) \equiv (\mathcal{B}, b) \equiv (\mathcal{B}, f(b)) \equiv (\mathcal{A}, H(f(b))) \equiv (\mathcal{A}, g(H(b))),$$

where $g := H \circ f \circ H^{-1} : H(C) \to A$. Since $\mathcal{A}$ is homogeneous, g can be extend to an automorphism g' of $\mathcal{A}$. But then, f can also be extended to the automorphism $f' := H^{-1} \circ g' \circ H$ of $\mathcal{B}$, which proves that $\mathcal{B}$ is homogeneous and concludes the proof. □

Theorem 3.21. *Let $\mathcal{L}$ be a language and $\mathcal{A}$ be a countable $\mathcal{L}$-structure. If $\mathcal{A}$ is the union of an elementary chain of homogeneous $\mathcal{L}$-structures, then $\mathcal{A}$ is homogeneous.*

Proof. Suppose that $\mathcal{A}$ is the union of an elementary chain $\langle \mathcal{A}_\alpha : \alpha < \beta \rangle$, where each $\mathcal{A}_\alpha$, for $\alpha < \beta$ is an homogeneous $\mathcal{L}$-structure. Since $\mathcal{A}$ is countable, we want to consider a finite subset $B = \{b_0, \ldots, b_{m-1}\} \subseteq A$ and a map $f : B \to A$ such that

$$(\mathcal{A}, b_0, \ldots, b_{m-1}) \equiv (\mathcal{A}, f(b_0), \ldots, f(b_{m-1})).$$

We will extend f to an automorphism of A. For that, let $a \in {}^\omega A$ be an enumeration. We will define recursively two sequences $\langle b_{m+k} : k < \omega \rangle$ and $\langle f(b_{m+k}) : k < \omega \rangle$ so that

1. for $k < \omega$:
$$\begin{cases} b_{m+k} = a_j \text{ if } k = 2j \text{ for some } j < \omega \\ f(b_{m+k}) = a_j \text{ if } k = 2j+1 \text{ for some } j < \omega. \end{cases}$$

2. for $k < \omega$:
$$(\mathcal{A}, b_0, \ldots, b_{m+k}) \equiv (\mathcal{A}, f(b_0), \ldots, f(b_{m+k})).$$

Take $k < \omega$ and suppose that b_l and $f(b_l)$ have been defined for all $0 \leqslant l \leqslant m-1+k$ in order to satisfy (1) and (2). Let us distinguish two cases:

- Suppose that $k = 2j$ for some $j < \omega$. Let $\alpha < \beta$ be so that $b_i, f(b_i) \in A_\alpha$ for $0 \leqslant i \leqslant m-1+k$. Observe that
$$\begin{aligned}(\mathcal{A}_\alpha, b_0, \ldots, b_{m-1+k}) &\overset{\mathcal{A}_\alpha \prec \mathcal{A}}{\equiv} (\mathcal{A}, b_0, \ldots, b_{m-1+k} \\ &\overset{\text{(2) of I.H.}}{\equiv} (\mathcal{A}, f(b_0), \ldots, f(b_{m-1+k})) \\ &\overset{\mathcal{A}_\alpha \prec \mathcal{A}}{\equiv} (\mathcal{A}_\alpha, f(b_0), \ldots, f(b_{m-1+k})).\end{aligned}$$
Since $\mathcal{A}_\alpha$ is homogeneous, there exists an automorphism h of $\mathcal{A}_\alpha$ whi extends $f : \{b_0, \ldots, b_{m-1+k}\} \to A : b_i \to f(b_i)$. We then define
$$b_{m+k} \coloneqq a_j \quad \text{and} \quad f(b_{m+k}) \coloneqq h(a_j).$$
This way, (1) and (2) are satisfied for $0 \leqslant l \leqslant m+k$.
- Suppose that $k = 2j+1$ for some $j < \omega$. Proceed as in last case to obtain an automorphism h extending $f : \{b_0, \ldots, b_{m-1+k}\} \to A : b_i \to f(b_i)$, and then define
$$b_{m+k} \coloneqq h^{-1}(a_j) \quad \text{and} \quad f(b_{m+k}) = a_j.$$

Let us check that the extension obtain of our original map $f : B \to A$ is an automorphism of $\mathcal{A}$.

Well-defined: Take $i < j < \omega$ and observe that
$$\begin{aligned}b_i = b_j &\Longrightarrow (\mathcal{A}, b_0, \ldots, b_j) \vDash c_i = c_j \\ &\overset{(2)}{\Longrightarrow} (\mathcal{A}, f(b_0), \ldots, f(b_j)) \vDash c_i = c_j \\ &\Longrightarrow f(b_i) = f(b_j).\end{aligned}$$

Bijection: Since $a \in {}^\omega A$ is an enumeration, f must be bijective.

Automorphism:

- Let $c \in \mathcal{L}$ be a constant symbol. We have $c^\mathcal{A} = a_i = b_{m+2i}$ for some $i < \omega$. Then,
$$\begin{aligned}(\mathcal{A}, b_0, \ldots, b_{m+2i}) \vDash c = c_{m+2i} &\overset{(2)}{\Longrightarrow} (\mathcal{A}, f(b_0), \ldots, f(b_{m+2i})) \vDash c = c_{m+2i} \\ &\Longrightarrow c^\mathcal{A} = f(b_{m+2i}) = f(c^\mathcal{A}).\end{aligned}$$
- Let $g \in \mathcal{L}$ be an n-ary function symbol and $a_{i_1}, \ldots, a_{i_n} \in A$, where $i_1, \ldots, i_n < \omega$. Note that $a_{i_j} = b_{m+2i_j}$ for $1 \leqslant j \leqslant n$, and $g^\mathcal{A}(a_{i_1}, \ldots, a_{i_n}) = a_i = b_{m+2i}$ for some $i < \omega$. Let $i^* \coloneqq \max\{i, i_1, \ldots, i_n\}$. Observe that
$$\begin{aligned}&(\mathcal{A}, b_0, \ldots, b_{2m+i^*}) \vDash g(c_{2m+i_1}, \ldots, c_{2m+i_n}) = c_{2m+i} \\ &\qquad\overset{(2)}{\Longrightarrow} (\mathcal{A}, f(b_0), \ldots, f(b_{2m+i^*})) \vDash g(c_{2m+i_1}, \ldots, c_{2m+i_n}) = c_{2m+i} \\ &\qquad\Longrightarrow g^\mathcal{A}(f(b_{m+2i_1}), \ldots, f(b_{2m+i_n})) = f(b_{m+2i}) \\ &\qquad\Longrightarrow g^\mathcal{A}(f(a_{i_1}), \ldots, f(a_{i_n})) = f(a_i) = f(g^\mathcal{A}(a_{i_1}, \ldots, a_{i_n})).\end{aligned}$$

- Let $R \in \mathcal{L}$ be an n-ary predicate symbol and $a_{i_1}, \dots, a_{i_n} \in A$, where $i_1, \dots, i_n < \omega$. Let $i^* := \max\{i_1, \dots, i_n\}$. Then,

$$\begin{aligned}(a_{i_1}, \dots, a_{i_n}) \in R^{\mathcal{A}} &\iff (b_{m+2i_1}, \dots, b_{m+2i_n}) \in R^{\mathcal{A}} \\ &\iff (\mathcal{A}, b_0, \dots, b_{2m+i^*}) \vDash R(c_{m+2i_1}, \dots, c_{m+2i_n}) \\ &\overset{(2)}{\iff} (\mathcal{A}, f(b_0), \dots, f(b_{2m+i^*})) \vDash R(c_{m+2i_1}, \dots, c_{m+2i_n}) \\ &\iff (f(b_{m+2i_1}), \dots, f(b_{m+2i_n})) \in R^{\mathcal{A}} \\ &\iff (f(a_{i_1}), \dots, f(a_{i_n})) \in R^{\mathcal{A}}.\end{aligned}$$

We have proved that the extended function f is indeed an automorphism of $\mathcal{A}$, which proves that $\mathcal{A}$ is homogeneous and concludes the proof. □

The following two results will be given without proof, but they can be found in [1]. Note that they are proved in the broader framework of a κ-class $\mathfrak{M}$. Here we use $\kappa = \aleph_0$, and look at an easier sub-case where the hypothesis of elements being in $\mathfrak{M}_{<\kappa}$ is replace by the supposition of being finitely generated. They are also stated using the notion of homoiogeneity, which is an alternative definition that ends up being equivalent to homogeneity.

Theorem 3.22. *Let $\mathcal{L}$ be a language and $\mathcal{A}$ be an $\mathcal{L}$-structure with* $\operatorname{card}(\mathcal{A}) = \aleph_0$. *Then $\mathcal{A}$ has an elementary extension $\mathcal{B}$ which is homogeneous and with* $\operatorname{card}(B) = \aleph_0$.

Theorem 3.23 (Isomorphism theorem)**.** *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be countable homogeneous $\mathcal{L}$-structures. If every finitely generated substructure $\mathcal{A}_0 \subseteq \mathcal{A}$ is isomorphic to another finitely generated substructure $\mathcal{B}_0 \subseteq \mathcal{B}$ and vice-versa, then $\mathcal{A} \cong \mathcal{B}$.*

3.4 Saturated structures

Definition 3.24. Let $\mathcal{L}$ be a language and α be an ordinal. We will use the notation $F_\alpha := \{\phi(x) \in \mathrm{Fml}_{\mathcal{L}_\alpha}\}$. Let $\Delta \subseteq F_\alpha$ be a set of formulae, $\mathcal{A}$ be an $\mathcal{L}$-structure and $a \in {}^{\alpha}A$.

The set Δ is *satisfiable in* $(\mathcal{A}, a)$ if

$$\exists a^* \in A, \forall \phi \in \Delta : (\mathcal{A}, a) \vDash \phi(a^*).$$

The set Δ is *finitely satisfiable in* $(\mathcal{A}, a)$ if

$$\forall \Delta_0 \subseteq \Delta \text{ finite, } \Delta_0 \text{ is satisfiable in } (\mathcal{A}, a).$$

Let κ be a cardinal. The structure $\mathcal{A}$ is *κ-saturated* if

$$\forall \alpha < \kappa, \forall a \in {}^{\alpha}A, \forall \Delta \subseteq F_\alpha : \Delta \text{ finitely satisfiable in } (\mathcal{A}, a) \implies \Delta \text{ satisfiable in } (\mathcal{A}, a).$$

Remark. As an immediate observation, we have that for cardinals $\lambda < \kappa$, being κ-saturated implies being λ-saturated. We also have the two following small properties.

Proposition 3.25. *Let $\mathcal{L}$ be a language and $\mathcal{A}$ be an $\mathcal{L}$-structure. If $\mathcal{A}$ is finite, then $\mathcal{A}$ is κ-saturated, for any cardinal number κ.*

Proof. Say that $A = \{a_1, \ldots, a_n\}$. Take a cardinal κ, an ordinal $\alpha < \kappa$, $a \in {}^{\alpha}A$ and $\Delta \subseteq F_\alpha$. Looking at the contrapositive, suppose that Δ is not satisfiable in $(\mathcal{A}, a)$, i.e.,

$$\forall a^* \in A, \exists \phi \in \Delta : (\mathcal{A}, a) \nvDash \phi(a^*).$$

In other words, for all $1 \leqslant i \leqslant n$, there exists $\phi_i \in \Delta$ for which $(\mathcal{A}, a) \vDash \neg\phi_i(a_i)$. Since the finite subset $\{\phi_1, \ldots, \phi_n\}$ of Δ is not satisfiable in $(\mathcal{A}, a)$, we have that Δ is not finitely satisfiable in $(\mathcal{A}, a)$. Therefore, $\mathcal{A}$ is κ-saturated, as desired. ☐

Proposition 3.26. *Let $\mathcal{L}$ be a language, $\mathcal{A}$ be an $\mathcal{L}$-structure and κ be a cardinal. If $\mathcal{A}$ is κ-saturated, then $\mathcal{A}$ is finite or* $\operatorname{card}(\mathcal{A}) \geqslant \kappa$.

Proof. Looking at the contrapositive, suppose $\mathcal{A}$ is infinite and of cardinality less than κ. In other words, with $\lambda \coloneqq \operatorname{card}(A)$, we have $\aleph_0 \leqslant \lambda < \kappa$. Take $a \in {}^{\lambda}A$, an enumeration of A and consider the following set of $\mathcal{L}_\lambda$-formulae:

$$\Delta \coloneqq \{\neg x = c_\alpha : \alpha < \kappa\}.$$

For a finite subset of Δ of cardinality $m \in \omega$ to be satisfiable in $(\mathcal{A}, a)$, one need at least $m+1$ distinct elements in A, which is always the case. Hence, Δ is finitely satisfiable in $(\mathcal{A}, a)$. But Δ is not satisfiable in $(\mathcal{A}, a)$ hence $\mathcal{A}$ is not κ-saturated. ☐

There is another useful proposition very close to the last one which uses the exact same proof strategy.

Proposition 3.27. *Let $\mathcal{L}$ be a language containing an unary predicate symbol W, κ be a cardinal and $\mathcal{A}$ be a κ-saturated $\mathcal{L}$-structure. Then, either $W^{\mathcal{A}}$ is finite or* $\operatorname{card}\left(W^{\mathcal{A}}\right) \geqslant \kappa$.

Proof. As in Proposition 3.26, suppose that $\aleph_0 \leqslant \lambda < \kappa$, where $\lambda \coloneqq \operatorname{card}(\mathcal{A})$. Take $a \in {}^{\lambda}(W^{\mathcal{A}})$, an enumeration of $W^{\mathcal{A}}$ and let

$$\Delta \coloneqq \{W(x) \wedge \neg x = c_\alpha : \alpha < \kappa\}.$$

The set Δ is finitely satisfiable but not satisfiable in $(\mathcal{A}, a)$, which proves that $\mathcal{A}$ is not κ-saturated and concludes the proof. ☐

We will now head towards the main theorems of this section, which will be of great use later on.

Lemma 3.28. *Let $\mathcal{L}$ be a language, κ be a cardinal with* $\operatorname{card}(\mathcal{L}) \leqslant \kappa$ *and $\mathcal{B}$ be an $\mathcal{L}$-structure of cardinality 2^κ. There exists an elementary extension $\mathcal{B}_1$ of $\mathcal{B}$, also of cardinality 2^κ and having the following property:*

$$\forall \alpha < \kappa^+, \forall a \in {}^\alpha B, \forall \Delta \subseteq F_\alpha : \Delta \textit{ finitely satisfiable in } (\mathcal{B}, a) \Longrightarrow \Delta \textit{ satisfiable in } (\mathcal{B}_1, a).$$

Proof. This time, we add two disjoint sets of new constant symbols:

$$\begin{aligned} \mathcal{L}_{2^\kappa} &= \mathcal{L} \cup \{c_\gamma \mid \gamma < 2^\kappa\}, \\ \mathcal{L}_{2^\kappa, 2^\kappa} &= \mathcal{L}_{2^\kappa} \cup \{d_\gamma \mid \gamma < 2^\kappa\}. \end{aligned}$$

The role of those new constants d_γ will be to show the existence of specific elements of B. Let $b \in {}^{(2^\kappa)}B$ be an enumeration of B, and let

$$S \coloneqq \{\Sigma \subseteq F_{2^\kappa} \mid \operatorname{card}(\Sigma) \leqslant \kappa \text{ and } \Sigma \text{ finitely satisfiable in } (\mathcal{B}, b)\}.$$

This set is at the core of the the strategy to obtain the desired property. Observe that the cardinality of S can be bounded:

$$\begin{aligned} \operatorname{card}(F_{2^\kappa}) &\leqslant \operatorname{card}(\mathrm{Fml}_{\mathcal{L}_{2^\kappa}}) = \operatorname{card}(\mathcal{L}_{2^\kappa}) \overset{\kappa \geqslant \operatorname{card}(\mathcal{L})}{=} 2^\kappa \\ \operatorname{card}(S) &\leqslant \operatorname{card}(\{\Sigma \subseteq F_{2^\kappa} \mid \operatorname{card}(\Sigma) \leqslant \kappa\}) \\ &\leqslant \sup\{(2^\kappa)^\nu \mid \nu \leqslant \kappa\} \\ &= 2^{\kappa \cdot \kappa} = 2^\kappa. \end{aligned}$$

Now, let $\{\Sigma_\gamma \mid \gamma < 2^\kappa\}$ be an enumeration of S and $\Lambda \coloneqq \mathrm{Th}_{\mathcal{L}_{2^\kappa}}(\mathcal{B}, b)$. For all $\gamma < 2^\kappa$, we substitute our constants d_γ into our sets Σ_γ to get an $\mathcal{L}_{2^\kappa, 2^\kappa}$-theory:

$$\Gamma_\gamma \coloneqq \Sigma_\gamma\left[\tfrac{d_\gamma}{x}\right] = \left\{\phi\left[\tfrac{d_\gamma}{x}\right] \;\middle|\; \phi \in \Sigma_\gamma\right\}.$$

We now aim to prove the consistency of

$$\Theta \coloneqq \Lambda \cup \bigcup_{\gamma < 2^\kappa} \Gamma_\gamma.$$

For that, take a finite subset $\Theta_0 \subseteq \Theta$. Thus there exists $k \in \mathbb{N}$ and $\gamma_0, \ldots, \gamma_k < 2^\kappa$ such that

$$\Theta_0 \subseteq \Lambda \cup \bigcup_{0 \leqslant i \leqslant k} \Gamma_{\gamma_i}.$$

For $0 \leqslant i \leqslant k$, note that

$$\Theta_0^i \coloneqq \{\phi \in \Sigma_{\gamma_i} \mid \phi(d_{\gamma_i}) \in \Gamma_{\gamma_i} \cap \Theta_0\}$$

is a finite subset of Σ_{γ_i}. As $\Sigma_{\gamma_i} \in S$, it is finitely satisfiable in $(\mathcal{B}, b)$ and hence Θ_0^i is satisfiable in $(\mathcal{B}, b)$. It means that there is some $b'_{\gamma_i} \in B$ such that for all $\phi \in \Theta_0^i : (\mathcal{B}, b) \vDash \phi(b'_{\gamma_i})$. So if we let b'_γ be any arbitrary element of B for $\gamma \notin \{\gamma_0, \ldots, \gamma_k\}$, it defines some $b' \in {}^{(2^\kappa)}B$, which can be used to interpret the constants $\{d_\gamma \mid \gamma < 2^\kappa\}$. And because of its construction, $(\mathcal{B}, b, b')$ is a model of Θ_0. Therefore, by the compactness theorem, we know that Θ has a model too, say

$$(\mathcal{B}_1, b_1, b_1').$$

Observe that

$$\begin{aligned}\operatorname{card}(\Theta) &= \operatorname{card}(\Lambda) + \operatorname{card}\left(\bigcup_{\gamma<2^\kappa} \Gamma_\gamma\right)\\ &\leqslant \operatorname{card}(\mathrm{Sent}_{\mathcal{L}_{2^\kappa}}) + \operatorname{card}(\{\phi(d_\gamma) \mid \phi \in F_{2^\kappa} \text{ and } \gamma < 2^\kappa\})\\ &\leqslant 2^\kappa + 2^\kappa \cdot 2^\kappa = 2^\kappa.\end{aligned}$$

Since Θ contains Λ, it contains in particular all the sentences $\neg c_\gamma = c_\delta$ for $\gamma, \delta < 2^\kappa$ as they are all true in $(\mathcal{B}, b)$, which implies that B_1 contains at least 2^κ elements. Therefore, by the downward Löwenheim-Skolem theorem, we are allowed without loss of generality to say that $\operatorname{card}(B_1) = 2^\kappa$. Moreover,

$$\forall \sigma \in \mathrm{Sent}_{\mathcal{L}_{2^\kappa}} : (\mathcal{B}, b) \vDash \sigma \implies \sigma \in \Lambda \implies (\mathcal{B}_1, b_1) \vDash \sigma,$$

which implies that $(\mathcal{B}, b) \equiv (\mathcal{B}_1, b_1)$. By Lemma 3.10, it implies that $\mathcal{B}$ is elementarily embeddable in $\mathcal{B}_1$. So without loss of generality (i.e. up to isomorphism) $\mathcal{B}_1$ is an elementary extension of $\mathcal{B}$.

We still have to check that $\mathcal{B}_1$ satisfies the desired property. Take $\alpha < \kappa^+$ and $a \in {}^\alpha B$. As $\alpha < \kappa^+ \leqslant 2^\kappa$, it can always be extended to an enumeration $b \in {}^{(2^\kappa)}B$. Take $\Delta \subseteq F_\alpha \subseteq F_{2^\kappa}$ and suppose that Δ is finitely satisfiable in $(\mathcal{B}, a)$, so in particular finitely satisfiable in $(\mathcal{B}, b)$. Observe that $\operatorname{card}(\Delta) \leqslant \operatorname{card}(\mathrm{Fml}_{\mathcal{L}_\alpha}) \leqslant \kappa$, which means that $\Delta \in S$ and that $\Delta = \Sigma_\gamma$ for some $\gamma < 2^\kappa$. Therefore, using the same notations as before:

$$\begin{aligned}(\mathcal{B}_1, b_1, b_1') \vDash \Theta &\implies (\mathcal{B}_1, b_1) \vDash \Sigma_\gamma(b_1'(d_\gamma))\\ &\implies (\mathcal{B}_1, a) \vDash \Sigma_\gamma(b_1'(d_\gamma)), \text{ since no other constant symbol is involved}\\ &\implies \Delta \text{ is satisfiable in } (\mathcal{B}_1, a) \text{ as desired.}\end{aligned}$$

□

Theorem 3.29. *Let $\mathcal{L}$ be a language, κ be a cardinal with* $\operatorname{card}(\mathcal{L}) \leqslant \kappa$ *and $\mathcal{A}$ be an infinite $\mathcal{L}$-structure. If* $\operatorname{card}(\mathcal{A}) \leqslant 2^\kappa$*, then $\mathcal{A}$ has a κ^+-saturated elementary extension of cardinality 2^κ.*

Proof. Let $\mathcal{A}$ be an infinite $\mathcal{L}$-structure with $\operatorname{card}(\mathcal{A}) \leqslant 2^\kappa$. We will inductively construct an elementary chain $\langle \mathcal{A}_\alpha : \alpha < \kappa^+ \rangle$ of $\mathcal{L}$-structures of cardinality 2^κ so that the union of this sequence will be the extension of $\mathcal{A}$ that we are looking for.

- First, let $\mathcal{A}_0$ be an elementary extension of $\mathcal{A}$ with $\operatorname{card}(\mathcal{A}_0) = 2^\kappa$ as given by Löwenheim-Skolem upwards.
- For a successor ordinal $\beta + 1 < \kappa^+$, let $\mathcal{A}_{\beta+1}$ be the elementary extension of $\mathcal{A}_\beta$ given by the previous lemma, which is a structure of cardinality 2^κ.

- For a limit ordinal $\alpha < \kappa^+$, let
$$\mathcal{A}_\alpha := \bigcup_{\beta<\alpha} \mathcal{A}_\beta.$$
By the theorem of union of chains, $\mathcal{A}_\alpha$ is an elementary extension of each $\mathcal{A}_\beta$ for $\beta < \alpha$. Note that
$$\text{card}(\mathcal{A}_\alpha) = \sup\{\text{card}(\mathcal{A}_\beta) : \beta < \alpha\} = \sup\{2^\kappa : \beta < \alpha\} = 2^\kappa.$$

By the theorem of union of chains, letting
$$\mathcal{A}' := \bigcup_{\alpha<\kappa^+} \mathcal{A}_\alpha$$
gives us an elementary extension of each element of the chain, so in particular $\mathcal{A} \prec \mathcal{A}_0 \prec \mathcal{A}'$. And as above, we have $\text{card}(\mathcal{A}') = 2^\kappa$.

Now, it is time to check that $\mathcal{A}'$ is κ^+-saturated. For that, take $\beta < \kappa^+$, $a \in {}^\beta A'$ and a set $\Delta \subseteq F_\beta$ which is finitely satisfiable in $(\mathcal{A}', a)$. We want to prove that Δ is satisfiable in $(\mathcal{A}', a)$.

Observe that for each $\alpha < \beta$, $a(\alpha) \in A' = \bigcup_{\gamma<\kappa^+} \mathcal{A}_\gamma$, thus there exists some $\gamma_\alpha < \kappa^+$ such that $a(\alpha) \in A_{\gamma_\alpha}$. Let $\gamma^* := \sup\{\gamma_\alpha : \alpha < \beta\}$. By Proposition 3.15, κ^+ is regular since it is a successor cardinal. Observe that $\gamma^* < \kappa+$, as if they were equal, we would have found a sequence $\langle\gamma_\alpha : \alpha < \beta\rangle$ indexed by $\beta < \kappa^+$ and of elements $< \kappa^+$ which is cofinal in κ^+ which would contradict its regularity. Therefore, for each $\alpha < \beta$, $a(\alpha) \in A_{\gamma_\alpha} \subseteq A_{\gamma^*}$ and thus $a \in {}^\beta A_{\gamma^*}$.

We have that Δ is finitely satisfiable in $(\mathcal{A}', a)$. So for each finite $\Delta_0 \subseteq \Delta$, there exists $a_0 \in A' = \bigcup_{\delta<\kappa^+} A_\delta$ which satisfies Δ_0 in $(\mathcal{A}', a)$ and thus there exists a minimal $\delta_0 < \kappa^+$ such that $a_0 \in A_{\delta_0}$. Let δ^* be the supremum of all those δ_0. Since there is at most $F_\beta^{<\omega}$ finite $\Delta_0 \subseteq \Delta$ and since $\text{card}\left(F_\beta^{<\omega}\right) = \text{card}(F_\beta) = \max\{\beta, \text{card}(\mathcal{L})\} < \kappa^+$, we have with the same reasoning as for γ^* that $\delta^* < \kappa^+$. Therefore, $\epsilon^* := \max\{\gamma^*, \delta^*\} < \kappa^+$.

Fix a finite $\Delta_0 \subseteq \Delta$. We know the existence of some $a_0 \in A_{\epsilon^*}$ such that for all $\phi \in \Delta_0$ we have $(\mathcal{A}', a) \vDash \phi(a_0)$ and since $\mathcal{A}_{\epsilon^*} \prec \mathcal{A}'$, it implies $(\mathcal{A}_{\epsilon^*}, a) \vDash \phi(a_0)$. Hence, Δ is finitely satisfiable in $(\mathcal{A}_{\epsilon^*}, a)$. And by the property given by the last lemma which was used in the construction of our elementary chain, we have that Δ is satisfiable in $(\mathcal{A}_{\epsilon^*+1}, a)$. Since $\mathcal{A}_{\epsilon^*+1} \prec \mathcal{A}'$, Δ is satisfiable in $(\mathcal{A}', a)$. This proves that $\mathcal{A}'$ is κ^+-saturated, which concludes the proof. □

Theorem 3.30. *Let $\mathcal{L}$ be a language, W be a new unary predicate symbol, $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures with $\mathcal{B} \prec \mathcal{A}$ and κ be a cardinal. If $\mathcal{L}_W$ denotes $\mathcal{L} \cup \{W\}$ and $(\mathcal{A}, B)$ is a κ-saturated $\mathcal{L}_W$-structure, then $\mathcal{B}$ is a κ-saturated $\mathcal{L}$-structure.*

Proof. Take $\alpha < \kappa$, $a \in {}^\alpha B$ and $\Delta \subseteq F_\alpha$ which is finitely satisfiable in $(\mathcal{B}, a)$. For a finite $\Delta_0 \subseteq \Delta$, let ϕ_{Δ_0} be the conjunction of its formulae. By hypothesis, there exists $b_0 \in B$ such that $(\mathcal{B}, a) \vDash \phi_{\Delta_0}(b_0)$.

Since $\mathcal{B} \prec \mathcal{A}$, by Lemma 3.9, we have $(\mathcal{B}, a) \prec (\mathcal{A}, a)$. Therefore, $(\mathcal{A}, a) \vDash \phi_{\Delta_0}(b_0)$ and hence $(\mathcal{A}, B, a) \vDash \exists x(W(x) \wedge \phi_{\Delta_0}(x))$. This proves that $\Delta \cup \{W(x)\}$ is finitely satisfiable in $(\mathcal{A}, B, a)$. Let $b_0 \in A$ be an element satisfying $\Delta \cup \{W(x)\}$ in $(\mathcal{A}, B, a)$, so in particular $b_0 \in W^{(\mathcal{A},B,a)} = B$.

Since $(\mathcal{B}, a) \prec (\mathcal{A}, a)$, we have for all $\phi \in \Delta$:

$$\begin{aligned}(\mathcal{A}, B, a) \vDash \phi(b_0) &\Longrightarrow (\mathcal{A}, a) \vDash \phi(b_0)\\ &\Longrightarrow (\mathcal{B}, a) \vDash \phi(b_0).\end{aligned}$$

Therefore, Δ is satisfiable in $(\mathcal{B}, a)$, which proves that $\mathcal{B}$ is κ-saturated and concludes the proof. □

Theorem 3.31. *Let $\mathcal{L}$ be a language, κ be a cardinal and $\mathcal{A}, \mathcal{B}$ be two elementarily equivalent κ-saturated $\mathcal{L}$-structures of cardinality κ. Then $\mathcal{A} \cong \mathcal{B}$.*

Proof. Let $a \in {}^{\kappa}A$ and $b \in {}^{\kappa}B$ be enumerations. We want to use the criterion of Theorem 3.11, i.e., to define two new enumerations $a' \in {}^{\kappa}A$ and $b' \in {}^{\kappa}B$, so that $(\mathcal{A}, a') \equiv_{\mathcal{L}_\kappa} (\mathcal{B}, b')$. They will be constructed by induction on $\alpha < \kappa$. Take $\alpha < \kappa$ and suppose that $a'(\beta), b'(\beta)$ have already been defined for all $\beta < \alpha$ so that $(\mathcal{A}, a'|_\beta) \equiv_{\mathcal{L}_\beta} (\mathcal{B}, b'|_\beta)$. Let γ be the largest ordinal $\leqslant \alpha$ which is either 0 or a limit ordinal. We distinguish two cases.

- If α is of the form $\gamma + 2n$ for some $n < \omega$, let $a'(\alpha) := a(\gamma + n)$. Let us use the following notations:

  $$\begin{aligned}T &= \mathrm{Th}_{\mathcal{L}_{\alpha+1}}(\mathcal{A}, a'|_{\alpha+1})\\ T' &= \{\text{formula of } T \text{ with all occurrences of } c_\alpha \text{ replace by } x\} \subseteq F_\alpha.\end{aligned}$$

  We will define $b'(\alpha)$ as an element satisfying T' in $(\mathcal{B}, b'|_\alpha)$. Take $\phi_1, \ldots, \phi_k \in T'$ and observe that

  $$\begin{aligned}(\mathcal{A}, a'|_\alpha) \vDash (\phi_1 \wedge \ldots \wedge \phi_k)(a'(\alpha)) &\Longrightarrow (\mathcal{A}, a'|_\alpha) \vDash \exists x(\phi_1 \wedge \ldots \wedge \phi_k)\\ &\overset{\text{I.H.}}{\Longrightarrow} (\mathcal{B}, b'|_\alpha) \vDash \exists x(\phi_1 \wedge \ldots \wedge \phi_k).\end{aligned}$$

  In other words, T' is finitely satisfiable in $(\mathcal{B}, b'|_\alpha)$. Since $\mathcal{B}$ is κ-saturated and $\alpha < \kappa$, T' is in fact satisfiable in $(\mathcal{B}, b'|_\alpha)$. Therefore, $b'(\alpha)$ can be chosen as an element of B which satisfies T' in $(\mathcal{B}, b'|_\alpha)$. With this choice, we get that $\mathrm{Th}_{\mathcal{L}_{\alpha+1}}(\mathcal{A}, a'|_{\alpha+1}) = \mathrm{Th}_{\mathcal{L}_{\alpha+1}}(\mathcal{B}, b'|_{\alpha+1})$, i.e., $(\mathcal{A}, a'|_{\alpha+1}) \equiv_{\mathcal{L}_{\alpha+1}} (\mathcal{B}, b'|_{\alpha+1})$.

- If α is of the form $\gamma + 2n + 1$ for some $n < \omega$, let $b'(\alpha) := b(\gamma + n)$ and proceed by doing an exact symmetry of the other case to define $a'(\alpha)$ so that $(\mathcal{A}, a'|_{\alpha+1}) \equiv_{\mathcal{L}_{\alpha+1}} (\mathcal{B}, b'|_{\alpha+1})$.

Observe that the alternating construction ensures that a', respectively b', contains the image of a, respectively of b, in its own image and thus is an enumeration. Now, take $\sigma \in \mathrm{Sent}_{\mathcal{L}_\kappa}$. As it has only finitely many symbols, we have $\sigma \in \mathrm{Sent}_{\mathcal{L}_{\alpha+1}}$ for some $\alpha < \kappa$. Therefore:

$$\begin{aligned}(\mathcal{A}, a') \vDash \sigma &\Longleftrightarrow (\mathcal{A}, a'|_{\alpha+1}) \vDash \sigma\\ &\overset{\text{construction}}{\Longleftrightarrow} (\mathcal{B}, b'|_{\alpha+1}) \vDash \sigma\\ &\Longleftrightarrow (\mathcal{B}, b') \vDash \sigma.\end{aligned}$$

This proves that $(\mathcal{A}, a'|_{\alpha+1}) \equiv_{\mathcal{L}_\kappa} (\mathcal{B}, b'|_{\alpha+1})$. By Theorem 3.11, we have that $\mathcal{A} \cong \mathcal{B}$ as desired. □

3.5 Special structures

Definition 3.32. Let $\mathcal{L}$ be a language and $\mathcal{A}$ be an $\mathcal{L}$-structure of cardinality κ. We say that $\mathcal{A}$ is *special* if for all cardinals $\lambda < \kappa$ there exists a λ^+-saturated $\mathcal{L}$-structure $\mathcal{A}_\lambda$ such that $\langle \mathcal{A}_\lambda : \lambda < \kappa \rangle$ is an elementary chain whose union is $\mathcal{A}$. This chain is called the *specializing chain* of $\mathcal{A}$.

The two following basic lemmas reveal a bit about what is behind the notion of special structures.

Lemma 3.33. *Let $\mathcal{L}$ be a language and $\mathcal{A}$ be a κ-saturated $\mathcal{L}$-structure of cardinality κ. Then $\mathcal{A}$ is special.*

Proof. Let us construct the specializing chain of $\mathcal{A}$. For each cardinal $\lambda < \kappa$, we have that $\lambda^+ \leqslant \kappa$ and, as $\mathcal{A}$ is κ-saturated, by the remark following Definition 3.24 it is also λ^+-saturated. Therefore $\mathcal{A}$ is special by being the union of $\langle \mathcal{A} : \lambda < \kappa \rangle$. □

Lemma 3.34. *Let $\mathcal{L}$ be a language and $\mathcal{A}$ be an $\mathcal{L}$-structure of cardinality κ^+. Then $\mathcal{A}$ is κ^+-saturated if and only if $\mathcal{A}$ is special.*

Proof. ($\Rightarrow$) This is the previous lemma.

($\Leftarrow$) Suppose $\mathcal{A}$ has a specializing chain $\langle \mathcal{A}_\lambda : \lambda < \kappa^+ \rangle$. It means that

$$\mathcal{A} = \bigcup_{\lambda < \kappa^+} \mathcal{A}_\lambda = \bigcup_{\lambda \leqslant \kappa} \mathcal{A}_\lambda = \mathcal{A}_\kappa,$$

which is κ^+-saturated by definition. □

Theorem 3.35. *Let $\mathcal{L}$ be a language, μ be a cardinal with* $\operatorname{card}(\mathcal{L}) \leqslant \mu$ *and* $\mu = 2^{<\mu}$ *and $\mathcal{A}$ be an $\mathcal{L}$-structure with* $\operatorname{card}(\mathcal{A}) < \mu$. *Then $\mathcal{A}$ has an elementary extension which is a special $\mathcal{L}$-structure of cardinality μ.*

Proof. We will distinct cases depending on the nature of μ.

- Suppose μ is a successor cardinal, say $\mu = \kappa^+$. Since

$$\kappa^+ = \mu = 2^{<\mu} = \sup\Big\{2^\lambda : \lambda < \mu\Big\} = \sup\Big\{2^\lambda : \lambda \leqslant \kappa\Big\} = 2^\kappa,$$

  we have by Theorem 3.29 that $\mathcal{A}$ has a κ^+-saturated elementary extension $\mathcal{B}$ of cardinality $2^\kappa = \kappa^+$. And by the previous lemma, $\mathcal{B}$ is special.

- Suppose μ is a limit cardinal. So in particular, for every cardinal $\kappa < \mu$, we also have $\kappa^+ < \mu$. Hence, if $\mathcal{A}$ is of cardinality κ, then

$$2^\kappa \leqslant \sup\Big\{2^\lambda : \lambda < \mu\Big\} = 2^{<\mu} = \mu.$$

We will inductively construct an elementary chain $\langle \mathcal{B}_\lambda : \lambda < \mu \rangle$ such that $\mathcal{B}_\lambda$ is λ^+-saturated for each $\lambda < \mu$, and $\operatorname{card}(\mathcal{B}_\lambda) \leqslant 2^\lambda$ for each $\kappa \leqslant \lambda < \mu$. The union of this chain will be our desired extension.

- By Theorem 3.29, $\mathcal{A}$ has a κ^+-saturated elementary extension $\mathcal{A}'$ of cardinality 2^κ. Thus for each $\lambda \leqslant \kappa$, let $\mathcal{B}_\lambda := \mathcal{A}'$.
- Suppose $\kappa < \lambda < \mu$ and λ is a successor cardinal, say $\lambda = \nu^+$. Observe that

$$\operatorname{card}(\mathcal{B}_\nu) \overset{\text{I.H.}}{\leqslant} 2^\nu \leqslant 2^{\nu^+}.$$

  Hence, let $\mathcal{B}_{\nu^+}$ be the ν^{++}-saturated elementary extension of $\mathcal{B}_\nu$ of cardinality 2^{ν^+} which is given by Theorem 3.29.
- Suppose $\kappa < \lambda < \mu$ and λ is a limit cardinal. By the theorem of union of chains, $\bigcup_{\nu<\lambda} \mathcal{B}_\nu$ is an elementary extension of each $\mathcal{B}_\nu$ with $\nu < \lambda$ and

$$\operatorname{card}\left(\bigcup_{\nu<\lambda} \mathcal{B}_\nu\right) \overset{\text{I.H.}}{\leqslant} \lambda \cdot 2^\lambda = 2^\lambda.$$

  Therefore, let $\mathcal{B}_\lambda$ be the λ^+-saturated elementary extension of $\bigcup_{\nu<\lambda} \mathcal{B}_\nu$ of cardinality 2^λ which is given by Theorem 3.29.

This inductive construction has given us $\langle \mathcal{B}_\lambda : \lambda < \mu \rangle$, which is the specializing chain of

$$\mathcal{B} := \bigcup_{\lambda<\mu} \mathcal{B}_\lambda.$$

Therefore, $\mathcal{B}$ is special and, by the theorem of union of chains, we have

$$\mathcal{A} \prec \mathcal{A}' = \mathcal{B}_\kappa \prec \mathcal{B},$$

which concludes the proof.

□

Corollary 3.36. *Let $\mathcal{L}$ be a language and μ be a limit beth number with* $\operatorname{card}(\mathcal{L}) < \mu$. *Then any $\mathcal{L}$-theory T with an infinite model has a special model of cardinality μ.*

Proof. By Löwenheim-Skolem, T has an infinite model of cardinality $< \mu$. And since $\mu = 2^{<\mu}$ as seen in Proposition 3.18, this models has, by the previous theorem, an elementary extension which is a special model of cardinality μ. □

The next proposition is a consequence of Proposition 3.27.

Proposition 3.37. *Let $\mathcal{L}$ be a language containing an unary predicate symbol W and $\mathcal{M}$ be an special $\mathcal{L}$-structure with $W^{\mathcal{M}}$ infinite. Then* $\operatorname{card}(\mathcal{M}) = \operatorname{card}\left(W^{\mathcal{M}}\right)$.

Proof. Let $\kappa := \operatorname{card}(\mathcal{M})$ and $\langle \mathcal{M}_\lambda : \lambda < \kappa \rangle$ be the specializing chain of $\mathcal{M}$. For each cardinal $\lambda < \kappa$, $\mathcal{M}_\lambda$ is an λ^+-saturated $\mathcal{L}$-structure, so by Proposition 3.27, either $\operatorname{card}\left(W^{\mathcal{M}_\lambda}\right) < \aleph_0$ or $\operatorname{card}\left(W^{\mathcal{M}_\lambda}\right) \geqslant \lambda^+$.

Since $W^{\mathcal{M}}$ is infinite, we have $\mathcal{M} \vDash \sigma_n$ for each $n < \omega$, where σ_n is the following $\mathcal{L}$-sentence:

$$\forall x_0 \ldots \forall x_n \exists x_{n+1} \left(W(x_0) \wedge \ldots \wedge W(x_n) \wedge W(x_{n+1}) \wedge \left(\bigwedge_{1 \leqslant i < j \leqslant n+1} \neg x_i = x_j \right) \right).$$

As the specializing chain is an elementary chain, have $\mathcal{M}_\lambda \prec \mathcal{M}$ and thus $\mathcal{M}_\lambda \vDash \sigma_n$, for each $\lambda < \kappa$ and $n < \omega$. Therefore, $\operatorname{card}\left(W^{\mathcal{M}_\lambda}\right) \geqslant \lambda^+$ for each $\lambda < \kappa$, and moreover

$$\begin{aligned} \kappa \geqslant \operatorname{card}\left(W^{\mathcal{M}}\right) &= \operatorname{card}\left(\bigcup_{\lambda<\kappa} W^{\mathcal{M}_\lambda} \right) \\ &\geqslant \operatorname{card}\left(\bigcup_{\lambda<\kappa} \lambda^+ \right) = \kappa. \end{aligned}$$

In other words, $\operatorname{card}(\mathcal{M}) = \kappa = \operatorname{card}\left(W^{\mathcal{M}}\right)$, which concludes the proof. □

Definition 3.38. Let $\mathcal{L}$ be a language. Consider a new unary predicate symbol W and let $\mathcal{L}_W := \mathcal{L} \cup \{W\}$. To each $\mathcal{L}$-formula ϕ, let us create an $\mathcal{L}_W$-formula, denoted $\phi^{(W)}$ and called the *relativization* of ϕ, as follows:

- If ϕ is atomic. let $\phi^{(W)}$ be ϕ.
- If ϕ is $\neg\psi$, let $\phi^{(W)}$ be $\neg\psi^{(W)}$.
- If ϕ is $\psi \wedge \theta$, let $\phi^{(W)}$ be $\psi^{(W)} \wedge \theta^{(W)}$.
- If ϕ is $\exists x \psi$, let $\phi^{(W)}$ be $\exists x(W(x) \wedge \psi^{(W)})$.

The idea of the relativization of an $\mathcal{L}$-formula ϕ is to express the same assertion about elements satisfying W as ϕ does with all elements.

The next theorem is a consequence of Theorem 3.30

Theorem 3.39. *Let $\mathcal{L}$ be a language, W be a new unary predicate symbol and $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures with $\mathcal{B} \prec \mathcal{A}$. If $\mathcal{L}_W$ denotes $\mathcal{L} \cup \{W\}$ and $(\mathcal{A}, B)$ is a special $\mathcal{L}_W$-structure, then $\mathcal{B}$ is a special $\mathcal{L}$-structure.*

Proof. Let $\kappa := \operatorname{card}(\mathcal{A})$. Suppose that $(\mathcal{A}, B)$ is special with specializing chain $\langle (\mathcal{A}_\lambda, B_\lambda) : \lambda < \kappa \rangle$. For each $\lambda < \kappa$, let $\mathcal{B}_\lambda := \mathcal{A}_\lambda|_{B_\lambda}$. We claim that $\langle \mathcal{B}_\lambda : \lambda < \kappa \rangle$ is a specializing chain for $\mathcal{B}$.

Note that we already have $B = W^{(\mathcal{A},B)} = \bigcup_{\lambda<\kappa} W^{(\mathcal{A}_\lambda, B_\lambda)} = B_\lambda$.

Fix $\lambda < \kappa$. Observe that B_λ is indeed the domain of $\mathcal{B}_\lambda$, since it is closed under the interpretations of function symbols. To see that, take an n-ary function symbol $f \in \mathcal{L}$ and $a_1, \ldots, a_n \in B_\lambda$, and let ϕ be the following $\mathcal{L}_W$-formula:

$$\exists x(W(x) \wedge f(x_1, \ldots, x_n) = x).$$

Then,

$$\begin{aligned} a_1, \ldots, a_n \in B_\lambda &\overset{B_\lambda \subseteq B}{\Longrightarrow} a_1, \ldots, a_n \in B \\ &\overset{\mathcal{B} \text{ is a structure}}{\Longrightarrow} f^{\mathcal{B}}(a_1, \ldots, a_n) \in B \\ &\Longrightarrow (\mathcal{B}, B) \vDash \phi(a_1, \ldots, a_n) \\ &\overset{\mathcal{B} \prec \mathcal{A} \Rightarrow (\mathcal{B}, B) \prec (\mathcal{A}, B)}{\Longrightarrow} (\mathcal{A}, B) \vDash \phi(a_1, \ldots, a_n) \\ &\overset{(\mathcal{A}_\lambda, B_\lambda) \prec (\mathcal{A}, B)}{\Longrightarrow} (\mathcal{A}_\lambda, B_\lambda) \vDash \phi(a_1, \ldots, a_n) \\ &\Longrightarrow f^{\mathcal{A}}(a_1, \ldots, a_n) = f^{\mathcal{A}_\lambda}(a_1, \ldots, a_n) \in B_\lambda. \end{aligned}$$

Secondly, by Theorem 3.30, $(\mathcal{A}_\lambda, B_\lambda)$ being λ^+-saturated implies that $\mathcal{B}_\lambda$ is λ^+-saturated.

Finally, fix $\lambda < \mu < \kappa$. What is left to show is that $\mathcal{B}_\lambda \prec \mathcal{B}_\mu$. Take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a_1, \ldots, a_n \in B_\lambda$. Let $\psi(x_1, \ldots, x_n)$ be the following $\mathcal{L}_W$-formula:

$$W(x_1) \wedge \ldots \wedge W(x_n) \wedge \phi^{(W)}(x_1, \ldots, x_n).$$

Then,

$$\begin{aligned} \mathcal{B}_\lambda \vDash \phi(a_1, \ldots, a_n) &\Longleftrightarrow (\mathcal{B}_\lambda, B_\lambda) \vDash \psi(a_1, \ldots, a_n) \\ &\overset{(*)}{\Longleftrightarrow} (\mathcal{A}_\lambda, B_\lambda) \vDash \psi(a_1, \ldots, a_n) \\ &\overset{(\mathcal{A}_\lambda, B_\lambda) \prec (\mathcal{A}, B)}{\Longleftrightarrow} (\mathcal{A}, B) \vDash \psi(a_1, \ldots, a_n) \\ &\overset{\mathcal{B} \prec \mathcal{A} \Rightarrow (\mathcal{B}, B) \prec (\mathcal{A}, B)}{\Longrightarrow} (\mathcal{B}, B) \vDash \psi(a_1, \ldots, a_n) \\ &\Longleftrightarrow \mathcal{B} \vDash \phi(a_1, \ldots, a_n). \end{aligned}$$

The step $(*)$ can be verified by a very quick induction on the complexity of ϕ, as it has already been made many times. The only case which normally gives a problem is the quantifier case, but here the relativization solves everything since it allows us to only consider elements in the interpretation of W. This proves that $\mathcal{B}_\lambda \prec \mathcal{B}_\mu$, and thus $\langle \mathcal{B}_\lambda : \lambda < \kappa \rangle$ is a specializing chain for $\mathcal{B}$, which concludes the proof. □

Theorem 3.40. *Let $\mathcal{L}$ be a language and $\mathcal{A}, \mathcal{B}$ be $\mathcal{L}$-structures. If $\mathcal{A}, \mathcal{B}$ are special, $\mathcal{A} \equiv \mathcal{B}$ and* $\mathrm{card}(\mathcal{A}) = \mathrm{card}(\mathcal{B})$*, then $\mathcal{A} \cong \mathcal{B}$.*

Proof. The case when $\mathcal{A}, \mathcal{B}$ are finite does not even require the structures to be special and has already been proved in Corollary 3.8. Suppose then that $\mathcal{A}, \mathcal{B}$ are infinite of cardinality κ. Let

$\langle \mathcal{A}_\lambda : \lambda < \kappa \rangle$ and $\langle \mathcal{B}_\lambda : \lambda < \kappa \rangle$ be their respective specializing chains. To show $\mathcal{A} \cong \mathcal{B}$, we will use the criterion of Theorem 3.11.

Define a_0 to be an enumeration of $\mathcal{A}_0$ and in general, a_λ to be an enumeration of $\mathcal{A}_\lambda$ extending all a_μ for $0 \leqslant \mu < \lambda \leqslant \kappa$. This defines an enumeration $a := a_\kappa \in {}^\kappa A$, which has the property that for $\alpha < \kappa$ with $\lambda := \operatorname{card}(\alpha)$, then $a(\alpha) = a_\lambda(\alpha) \in A_\lambda$. Similarly, we can define an enumeration $b \in {}^\kappa B$ such that for all $\alpha < \kappa$ with $\lambda := \operatorname{card}(\alpha)$, then $b(\alpha) \in B_\lambda$.

We will define two new enumerations $a' \in {}^\kappa A$ and $b' \in {}^\kappa B$ such that for all $\alpha < \kappa$ with $\lambda := \operatorname{card}(\alpha)$, the following conditions are fulfilled:

1. $a'(\alpha) \in A_\lambda$, $b'(\alpha) \in B_\lambda$.
2. $(\mathcal{A}, a'|_{\alpha+1}) \equiv_{\mathcal{L}_{\alpha+1}} (\mathcal{B}, b'|_{\alpha+1})$.
3. $a'(\alpha) = a(\beta + n)$ if $\alpha = \beta + 2n$ for some $n < \omega$.
   $b'(\alpha) = b(\beta + n)$ if $\alpha = \beta + 2n + 1$ for some $n < \omega$.

Fix $\alpha < \kappa$ with $\lambda := \operatorname{card}(\alpha)$ such that $a'(\beta)$ and $b'(\beta)$ have already been defined for all $\beta < \alpha$ such that so far (1), (2), (3) have been satisfied.

If α is a successor ordinal, say $\beta + 1$, then

$$(\mathcal{A}, a'|_\alpha) = (\mathcal{A}, a'|_{\beta+1}) \overset{(2)}{\equiv}_{\mathcal{L}_{\beta+1}} (\mathcal{B}, b'|_{\beta+1}) = (\mathcal{B}, b'|_\alpha).$$

If α is a limit ordinal, then for all $\sigma \in \mathrm{Sent}_{\mathcal{L}_\alpha}$, there exists $\beta < \alpha$ such that $\sigma \in \mathrm{Sent}_{\mathcal{L}_{\beta+1}}$ and thus

$$\begin{aligned}(\mathcal{A}, a'|_\alpha) \vDash \sigma &\iff (\mathcal{A}, a'|_{\beta+1}) \vDash \sigma \\ &\overset{(2)}{\iff} (\mathcal{B}, b'|_{\beta+1}) \vDash \sigma \\ &\iff (\mathcal{B}, b'|_\alpha) \vDash \sigma.\end{aligned}$$

Therefore, we have $(\mathcal{A}, a'|_\alpha) \equiv_{\mathcal{L}_\alpha} (\mathcal{B}, b'|_\alpha)$ in all cases.

By (1), we have $a'|_\alpha \in {}^\alpha(A_\lambda)$ and $b'|_\alpha \in {}^\alpha(B_\lambda)$. Since $\mathcal{A}_\lambda \prec \mathcal{A}$ and $\mathcal{B}_\lambda \prec \mathcal{B}$, Lemma 3.9 implies that $(\mathcal{A}_\lambda, a'|_\alpha) \prec (\mathcal{A}, a'|_\alpha)$ and $(\mathcal{B}_\lambda, b'|_\alpha) \prec (\mathcal{B}, b'|_\alpha)$. Therefore,

$$(\mathcal{A}_\lambda, a'|_\alpha) \equiv_{\mathcal{L}_\alpha} (\mathcal{A}, a'|_\alpha) \equiv_{\mathcal{L}_\alpha} (\mathcal{B}, b'|_\alpha) \equiv_{\mathcal{L}_\alpha} (\mathcal{B}_\lambda, b'|_\alpha).$$

On another note, recall that $\mathcal{A}_\lambda$ and $\mathcal{B}_\lambda$ are λ^+-saturated. Together with condition (3), this is the exact same construction that has been done in Theorem 3.31, hence the argument can be repeated here to find elements $a'(\alpha)$ and $b'(\alpha)$ such that

$$(\mathcal{A}, a'|_{\alpha+1}) \equiv_{\mathcal{L}_{\alpha+1}} (\mathcal{B}, b'|_{\alpha+1}).$$

This finishes the inductive proof of a' and b'. By condition (3), we made sure to have them surjective, just as a, b were, and they are therefore also enumerations.

For all $\alpha < \kappa$, we have $(\mathcal{A}, a'|_{\alpha+1}) \equiv_{\mathcal{L}_{\alpha+1}} (\mathcal{B}, b'|_{\alpha+1})$, which implies that $(\mathcal{A}, a') \equiv_{\mathcal{L}_\kappa} (\mathcal{B}, b')$, and thus, Theorem 3.11 tells us that $\mathcal{A} \cong \mathcal{B}$. $\square$

3.6 Vaught Theorem

Definition 3.41. Let $\mathcal{L}$ be a language containing an unary predicate symbol U, T be an $\mathcal{L}$-theory and κ, λ be cardinals. We say that T *admits* the pair of cardinals $\langle \kappa, \lambda \rangle$ if it has a model $\mathcal{M}$ with $\operatorname{card}(\mathcal{M}) = \kappa$ and $\operatorname{card}\left(U^{\mathcal{M}}\right) = \lambda$.

The main question of interest of this section will be: When we know T to admits a pair of cardinals, which other pairs does it also admit?

Theorem 3.42. *Let $\mathcal{L}$ be a language containing an unary predicate symbol U, T be an $\mathcal{L}$-theory and κ, λ, ν be infinite cardinals.*

- *If T admits $\langle \kappa, \lambda \rangle$ and $\nu \geqslant \operatorname{card}(T)$, then it admits $\langle \nu, \nu \rangle$.*
- *If T admits $\langle \kappa, \lambda \rangle$ and $\kappa \geqslant \nu \geqslant \max\{\lambda, \operatorname{card}(T), \operatorname{card}(\mathcal{L})\}$, then it admits $\langle \nu, \lambda \rangle$.*

Proof. Suppose T admits $\langle \kappa, \lambda \rangle$, i.e., it has a model $\mathcal{M}$ with $\operatorname{card}(\mathcal{M}) = \kappa$ and $\operatorname{card}\left(U^{\mathcal{M}}\right) = \lambda$. Let

$$\begin{aligned} \mathcal{L}_\nu &:= \mathcal{L} \cup \{c_\alpha : \alpha < \nu\}, \\ T' &:= T \cup \{\neg c_\alpha = c_\beta : \alpha < \beta < \nu\} \cup \{U(c_\alpha) : \alpha < \nu\}, \end{aligned}$$

where the c_α for $\alpha < \nu$ are new constant symbols. Since $\operatorname{card}\left(U^{\mathcal{M}}\right) = \lambda$ and $\mathcal{M} \vDash T$, observe that any finite subset of T' has a model and thus, by the compactness theorem, T' itself has a model.

- Suppose that $\nu \geqslant \operatorname{card}(T)$. Thus $\operatorname{card}(T') = \nu$ and Löwenheim-Skolem theorem implies that T' has a model $\mathcal{N}$ of cardinality ν. Observe that

  $$\nu \geqslant \operatorname{card}\left(U^{\mathcal{N}}\right) \geqslant \operatorname{card}\left(\left\{c_\alpha^{\mathcal{N}} : \alpha < \nu\right\}\right) = \nu,$$

  hence T admits $\langle \nu, \nu \rangle$.
- Suppose that $\kappa \geqslant \nu \geqslant \max\{\lambda, \operatorname{card}(T), \operatorname{card}(\mathcal{L})\}$. By Theorem 3.5, there is an $\mathcal{L}$-structure $\mathcal{N}'$ of cardinality ν such that $\mathcal{M}|_{U^{\mathcal{M}}} \subseteq \mathcal{N}' \prec \mathcal{M}$. In particular, $U^{\mathcal{M}} \subseteq U^{\mathcal{N}'} \subseteq U^{\mathcal{M}}$, and thus $\operatorname{card}\left(U^{\mathcal{N}'}\right) = \lambda$. Therefore, T admits $\langle \nu, \lambda \rangle$.

□

Theorem 3.43. *Let $\mathcal{L}$ be a language containing an unary predicate symbol U, T be an $\mathcal{L}$-theory and κ, λ, μ be infinite cardinals. If T admits $\langle \kappa, \lambda \rangle$, then it admits $\langle \kappa^\mu, \lambda^\mu \rangle$.*

Proof. Let $\mathcal{M}$ be a model of T with $\operatorname{card}(\mathcal{M}) = \kappa$, $\operatorname{card}\left(U^{\mathcal{M}}\right) = \lambda$ and μ be an arbitrary infinite cardinal. By Lemma 2.25, there exists a regular ultrafilter $\mathcal{U}$ on μ. Consider the ultraproduct

$$\mathcal{N} := {}^{\mu}\mathcal{M} \,/\, \mathcal{U},$$

which, by Lemma 2.27 and Theorem 2.31, is of cardinality κ^μ. By Corollary 2.33, $\mathcal{N}$ is also a model of T. The only detail left is to check the cardinality of $U^{\mathcal{N}}$.

$$\begin{aligned} U^{\mathcal{N}} &\overset{\text{def ultraproduct}}{=} \left\{[f] \mid f : \mu \to M \text{ and } \left\{i \in \mu : f(i) \in U^{\mathcal{M}}\right\} \in \mathcal{U}\right\} \\ &\overset{(*)}{=} \left\{[f] \mid f : \mu \to U^{\mathcal{M}}\right\} \\ &= {}^{\mu}(U^{\mathcal{M}}) \,/\, \mathcal{U}. \end{aligned}$$

The second equality $(*)$ comes from the fact that for each $f : \mu \to M$ with $\left\{i \in \mu : f(i) \in U^{\mathcal{M}}\right\} \in \mathcal{U}$, we can define

$$g : \mu \to U^{\mathcal{M}} : i \mapsto \begin{cases} f(i) \text{ if } f(i) \in U^{\mathcal{M}} \\ \text{a fixed element of } U^{\mathcal{M}} \text{ otherwise,} \end{cases}$$

which is equivalent to f since

$$\{i \in I : f(i) = g(i)\} \supseteq \left\{i \in I : f(i) \in U^{\mathcal{M}}\right\} \in \mathcal{U}.$$

Therefore,

$$\operatorname{card}\left(U^{\mathcal{N}}\right) = \operatorname{card}\left({}^{\mu}(U^{\mathcal{M}}) \,/\, \mathcal{U}\right) \overset{\text{Theorem 2.31}}{=} \operatorname{card}\left(U^{\mathcal{M}}\right)^{\mu} = \lambda^\mu,$$

which concludes the proof. □

We now aim to prove Vaught theorem, which states the following.

Theorem 3.44 (Vaught Theorem)**.** *Let $\mathcal{L}$ be a countable language containing an unary predicate symbol U, T be an $\mathcal{L}$-theory and κ, λ be cardinals with $\aleph_0 \leqslant \lambda < \kappa$. If T admits $\langle \kappa, \lambda \rangle$, then it admits $\langle \aleph_1, \aleph_0 \rangle$.*

To prove it, we will need to add another unary predicate symbol to our language and study a specific variant of the theory which is in the extended language.

Definition 3.45. Let $\mathcal{L}$ be a language. Consider a new unary predicate symbol W and let $\mathcal{L}_W \coloneqq \mathcal{L} \cup \{W\}$. For each $\mathcal{L}$-formula ϕ, let us recall how the relativization $\phi^{(W)} \in \mathrm{Fml}_{\mathcal{L}_W}$ of ϕ is defined:

- If ϕ is atomic. let $\phi^{(W)}$ be ϕ.
- If ϕ is $\neg\psi$, let $\phi^{(W)}$ be $\neg\psi^{(W)}$.
- If ϕ is $\psi \wedge \theta$, let $\phi^{(W)}$ be $\psi^{(W)} \wedge \theta^{(W)}$.
- If ϕ is $\exists x\psi$, let $\phi^{(W)}$ be $\exists x \left(W(x) \wedge \psi^{(W)}\right)$.

The idea of the relativization of an $\mathcal{L}$-formula ϕ is to express the same assertion about elements satisfying W as ϕ does with all elements. Suppose now that $\mathcal{L}$ contains another unary predicate symbol $\mathcal{U}$. For each $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$, define a corresponding $\mathcal{L}_W$-sentence ϕ_W as:

$$\forall x_1 \ldots \forall x_n \left(W(x_1) \wedge \ldots \wedge W(x_n) \to \left(\phi \leftrightarrow \phi^{(W)}\right)\right).$$

For each $\mathcal{L}$-theory T, define a corresponding $\mathcal{L}_W$-theory as such:

$$T_W \coloneqq T \cup \{\phi_W : \phi \in \mathrm{Fml}_{\mathcal{L}}\} \cup \{\forall x(U(x) \to W(x)),\ \exists x \neg W(x)\}.$$

Lemma 3.46. *Let $\mathcal{L}$ be a countable language containing an unary predicate symbol U, W be a new unary predicate symbol, T be an $\mathcal{L}$-theory and κ, λ be cardinals with $\aleph_0 \leqslant \lambda < \kappa$. If T admits $\langle \kappa, \lambda \rangle$, then T_W is consistent.*

Proof. Let $\mathcal{M}$ be a model of T with $\mathrm{card}(\mathcal{M}) = \kappa$ and $\mathrm{card}\left(U^{\mathcal{M}}\right) = \lambda$. By Theorem 3.5, there exists an $\mathcal{L}$-structure $\mathcal{N}$ with $\mathrm{card}(\mathcal{N}) = \lambda$ and

$$\mathcal{M}|_{U^{\mathcal{M}}} \subseteq \mathcal{N} \prec \mathcal{M}.$$

We want to prove that the $\mathcal{L}_W$-structure $(\mathcal{M}, N)$ models T_W.

- First, $\mathcal{M} \vDash T$ implies that $(\mathcal{M}, N) \vDash T$.
- Second, since $\mathcal{M}|_{U^{\mathcal{M}}} \subseteq \mathcal{N}$, we have $U^{(\mathcal{M},N)} = U^{\mathcal{M}} \subseteq N = W^{(\mathcal{M},N)}$ and thus

$$(\mathcal{M}, N) \vDash \forall x(U(x) \to W(x)).$$

- Third, as $\lambda < \kappa$, then $N \subsetneq M$ and hence $(\mathcal{M}, N) \vDash \exists x \neg W(x)$.
- Finally, take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a_1, \ldots, a_n \in N = W^{(\mathcal{M},N)}$. Then, we have

$$(\mathcal{M}, N) \vDash \phi(a_1, \ldots, a_n) \Leftrightarrow (\mathcal{M}, N) \vDash \phi^{(W)}(a_1, \ldots, a_n),$$

  which can be seen by a very quick induction on the complexity of ϕ which follows immediatly from the fact that $\mathcal{N} \prec \mathcal{M}$ and $a_1, \ldots, a_n \in W^{(\mathcal{M},N)}$. Therefore, $(\mathcal{M}, N) \vDash \phi_W$.

We have shown that $(\mathcal{M}, N) \vDash T_W$, which concludes the proof. □

Lemma 3.47. *Let $\mathcal{L}$ be a countable language containing an unary predicate symbol U, W be a new unary predicate symbol and T be an $\mathcal{L}$-theory. If T_W is consistent, then T admits $\langle \aleph_1, \aleph_0 \rangle$.*

Proof. Suppose that T_W is consistent, which means that it has a model. Observe that any model $\mathcal{M}$ of T_W must be infinite. Indeed, suppose ab absurdum that $M = \{a_1, \ldots, a_n\}$. We know that $k \coloneqq \mathrm{card}\left(W^{\mathcal{M}}\right) < n$. Suppose without loss of generality that $W^{\mathcal{M}} = \{a_1, \ldots, a_k\}$. Let $\phi \in \mathrm{Fml}_{\mathcal{L}}$ be

$$\exists x(\neg x = x_1 \wedge \ldots \wedge \neg x = x_k).$$

Since $\mathcal{M} \vDash \phi_W$ and $\mathcal{M} \vDash \phi(a_1, \ldots, a_k)$, then $\mathcal{M} \vDash \phi^{(W)}(a_1, \ldots, a_k)$, which contradicts $k = \mathrm{card}\left(W^{\mathcal{M}}\right)$ $\lightning$.

By Corollary 3.36, if μ is any limit Beth number with $\mu > \mathrm{card}(\mathcal{L})$, then we know that T_W has a special model $(\mathcal{M}, W_{\mathcal{M}})$ of cardinality μ.

We claim that $W_{\mathcal{M}}$ is closed under the interpretations of function symbols. Indeed, take an n-ary function symbol $f \in \mathcal{L}$ and $a_1, \dots, a_n \in W_{\mathcal{M}}$. Let $\phi \in \mathrm{Fml}_{\mathcal{L}}$ be

$$\exists x f(x_1, \dots, x_n) = x.$$

Since $\mathcal{M} \vDash \phi_W$ and $\mathcal{M} \vDash \phi(a_1, \dots, a_n)$, then $\mathcal{M} \vDash \phi^{(W)}(a_1, \dots, a_n)$, i.e., $f^{\mathcal{M}}(a_1, \dots, a_n) \in W_{\mathcal{M}}$. Therefore, the universe of $\mathcal{M}|_{W_{\mathcal{M}}}$ is $W_{\mathcal{M}}$.

Take $\phi(x_1, \dots, x_n) \in \mathrm{Fml}_{\mathcal{L}}$ and $a_1, \dots, a_n \in W_{\mathcal{M}}$. Observe that

$$\begin{aligned} \mathcal{M}|_{W_{\mathcal{M}}} \vDash \phi(a_1, \dots, a_n) &\iff \mathcal{M}|_{W_{\mathcal{M}}} \vDash \phi^{(W)}(a_1, \dots, a_n) \\ &\overset{(*)}{\iff} \mathcal{M} \vDash \phi^{(W)}(a_1, \dots, a_n) \\ &\overset{\mathcal{M} \vDash \phi_W}{\iff} \mathcal{M} \vDash \phi(a_1, \dots, a_n). \end{aligned}$$

The step $(*)$ holds since everything happens in the interpretation of W, this is the same argument as in the proof of Theorem 3.39. Therefore, $\mathcal{M}|_{W_{\mathcal{M}}} \prec \mathcal{M}$, and since $\mathcal{M}$ is infinite, $W_{\mathcal{M}}$ is infinite too. By Proposition 3.37, we have

$$\mathrm{card}\,(\mathcal{M}) = \mathrm{card}\,((\mathcal{M}, W_{\mathcal{M}})) = \mathrm{card}\,(W_{\mathcal{M}})\,.$$

By Theorem 3.39, $\mathcal{M}|_{W_{\mathcal{M}}}$ is also a special $\mathcal{L}$-structure. Therefore, by Theorem 3.40, $\mathcal{M}|_{W_{\mathcal{M}}} \cong \mathcal{M}$.

Let $H_{\mathcal{M}} : \mathcal{M} \cong \mathcal{M}|_{W_{\mathcal{M}}}$ be an isomorphism, R be a new binary predicate symbol, $\mathcal{L}_{W,R} \coloneqq \mathcal{L}_W \cup \{R\}$ and $(\mathcal{M}, W_{\mathcal{M}}, R_{\mathcal{M}})$ be the $\mathcal{L}_{W,R}$-structure with $R_{\mathcal{M}}$ being the graph of $H_{\mathcal{M}}$.

Consider $\Delta \coloneqq \mathrm{Th}_{\mathcal{L}_{W,R}}(\mathcal{M}, W_{\mathcal{M}}, R_{\mathcal{M}})$. Since Δ has an infinite model, by downward Löwenheim-Skolem and Theorem 3.22, it also has an homogeneous model of cardinality $\aleph_0$, say

$$(\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}).$$

Observe that $(\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}})$ being homogeneous implies that $\mathcal{N}$ is homogeneous. Indeed, take $n < \omega$, $B = \{b_0, \dots, b_n\} \subseteq N$ and $f : B \to N$ such that $(\mathcal{N}, b) \equiv (\mathcal{N}, f(b))$, where $b : n \to N : m \mapsto b_m$ and $f(b) : n \to N : m \mapsto f(b_m)$. Then in particular $(\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}, b) \equiv (\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}, f(b))$. The homogeneity of $(\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}})$ tells us that f can be extended to an automorphism of $\mathcal{N}$.

Since $(\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}) \vDash \Delta$ and $T_W \subseteq \Delta$, we have, as before for $W_{\mathcal{M}}$, that $W_{\mathcal{N}}$ is closed under the interpretations of function symbols and that $\mathcal{N}|_{W_{\mathcal{N}}} \prec \mathcal{N}$. Observe that $\mathcal{N}|_{W_{\mathcal{N}}}$ is a proper substructure of $\mathcal{N}$, i.e. $W_{\mathcal{N}} \subsetneq N$, since $\mathcal{N} \vDash \exists x \neg W(x)$. Note also that

$$U^{\mathcal{N}|_{W_{\mathcal{N}}}} = U^{\mathcal{N}} \cap W_{\mathcal{N}} = U^{\mathcal{N}}.$$

We now want to prove that $R_{\mathcal{N}} : \mathcal{N} \cong \mathcal{N}|_{W_{\mathcal{N}}}$. First, observe that

$$(\mathcal{M}, W_{\mathcal{M}}, R_{\mathcal{M}}) \vDash \forall x \exists! y \big(W(y) \land R(x,y)\big) \land \forall y \big(W(y) \to \exists! x R(x,y)\big).$$

Thus,

$$(\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}) \vDash \forall x \exists! y\big(W(y) \wedge R(x,y)\big) \wedge \forall y\big(W(y) \to \exists! x R(x,y)\big).$$

Therefore, $R_{\mathcal{N}}$ can be viewed as the graph of a bijective function $H_{\mathcal{N}} : \mathcal{N} \to \mathcal{N}|_{W_{\mathcal{N}}}$. Let us check the three conditions for it to be an isomorphism.

- Let $c \in \mathcal{L}$ be a constant symbol. Observe that

$$\begin{aligned} H_{\mathcal{M}}(c^{\mathcal{M}}) = c^{\mathcal{M}|_{W_{\mathcal{M}}}} &\implies (\mathcal{M}, W_{\mathcal{M}}, R_{\mathcal{M}}) \vDash R(c,c) \\ &\implies (\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}) \vDash R(c,c) \\ &\implies H(c^{\mathcal{N}}) = c^{\mathcal{N}|_{W_{\mathcal{N}}}}. \end{aligned}$$

- Let $f \in \mathcal{L}$ be an n-ary function symbol and $a_1, \ldots, a_n \in N$. For all $b_1, \ldots, b_n \in M$, we have $H_{\mathcal{M}}(f^{\mathcal{M}}(b_1, \ldots, b_n)) = f^{\mathcal{M}}(H(b_1), \ldots, H(b_n))$. Hence, if $\sigma \in \mathrm{Sent}_{\mathcal{L}_{W,R}}$ is defined as

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_n\big(R(x_1, y_1) \wedge \ldots \wedge R(x_n, y_n) \wedge R(f(x_1, \ldots, x_n), f(y_1, \ldots, y_n))\big),$$

  then

$$(\mathcal{M}, W_{\mathcal{M}}, R_{\mathcal{M}}) \vDash \sigma \implies (\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}) \vDash \sigma.$$

  Therefore, $H_{\mathcal{N}}\left(f^{\mathcal{N}}(a_1, \ldots, a_n)\right) = f^{\mathcal{N}}\big(H(a_1), \ldots, H(a_n)\big)$.

- Let $P \in \mathcal{L}$ be an n-ary predicate symbol and $a_1, \ldots, a_n \in N$. For all $b_1, \ldots, b_n \in M$, we have

$$(b_1, \ldots, b_n) \in P^{\mathcal{M}} \iff \big(H_{\mathcal{M}}(b_1), \ldots, H_{\mathcal{M}}(b_n)\big) \in P^{\mathcal{M}}.$$

  Hence, if $\sigma \in \mathrm{Sent}_{\mathcal{L}_{W,R}}$ is defined as

$$\forall x_1 \ldots \forall x_n \exists y_1 \ldots \exists y_n\big(R(x_1, y_1) \wedge \ldots \wedge R(x_n, y_n) \wedge (P(x_1, \ldots, x_n) \leftrightarrow P(y_1, \ldots, y_n))\big),$$

  then,

$$(\mathcal{M}, W_{\mathcal{M}}, R_{\mathcal{M}}) \vDash \sigma \implies (\mathcal{N}, W_{\mathcal{N}}, R_{\mathcal{N}}) \vDash \sigma.$$

  Therefore,

$$(a_1, \ldots, a_n) \in P^{\mathcal{N}} \iff \big(H_{\mathcal{M}}(a_1), \ldots, H_{\mathcal{M}}(a_n)\big) \in P^{\mathcal{M}},$$

So $H_{\mathcal{N}}$ is indeed an isomorphism. Since $\mathcal{N}$ is homogeneous, by Lemma 3.20 we also have that $\mathcal{N}|_{W_{\mathcal{N}}}$ is homogeneous. From now, we will use the notations $\mathcal{N}_0 \coloneqq \mathcal{N}|_{W_{\mathcal{N}}}$, $\mathcal{N}_1 \coloneqq \mathcal{N}$, and $H_0 \coloneqq H_{\mathcal{N}}^{-1}$. To summarize, we know that

1. $\mathcal{N}_0 \prec \mathcal{N}_1$ and $N_0 \subsetneq N_1$.
2. $H_0 : \mathcal{N}_0 \cong \mathcal{N}_1$ and they are both homogeneous.
3. $U^{\mathcal{N}_0} = U^{\mathcal{N}_1}$.

We want to define an elementary chain $\langle \mathcal{N}_\alpha : \alpha < \omega_1 \rangle$. Take $\alpha < \omega$ and let us distinguish cases.

If α is 0 or 1, $\mathcal{N}_\alpha$ has already been defined.

If α is a double successor ordinal, say $\alpha = \beta + 2$, we want to define $\mathcal{N}_{\beta+2}$ in relation to $\mathcal{N}_{\beta+1}$ in the same way that $\mathcal{N}_1$ is related to $\mathcal{N}_0$. Let us explicit it once by constructing $\mathcal{N}_2$. Observe that we cannot do the same procedure as in the beginning of the lemma, since the structures that we have and that we are constructing are no longer guaranteed to be special. Let X_1 be a bijective copy of $N_1 \setminus N_0$, with bijection $f : N_1 \setminus N_0 \approx X_1$. This notation will allow us to have a better readability. Define $N_2 \coloneqq N_1 \sqcup X_1$, this way we have $N_1 \subsetneq N_2$. Define

$$H_1 : N_1 \to N_2 : x \mapsto \begin{cases} H_0(x) \text{ if } x \in N_0 \\ f(x) \text{ if } N_1 \setminus N_0. \end{cases}$$

Schematically:

$$\begin{array}{ccc} N_0 & \xrightarrow[\cong]{H_0} & N_1 \\ \vdots & & \vdots \\ N_1 = N_0 \sqcup (N_1 \setminus N_0) & \overset{H_1}{\dashrightarrow} & N_2 = N_1 \sqcup X_1 \\ \vdots & & \vdots \\ N_1 \setminus N_0 & \xrightarrow[\approx]{f} & X_1 \end{array}$$

Observe that H_1 is a bijection. We want to construct $\mathcal{N}_2$ with universe N_2 so that H_1 is an isomorphism.

- For $c \in \mathcal{L}$, a constant symbol, let
$$c^{\mathcal{N}_2} \coloneqq H_1\left(c^{\mathcal{N}_1}\right).$$
- For $g \in \mathcal{L}$, an n-ary function symbol, and $a_1, \ldots, a_n \in N_2$, let
$$g^{\mathcal{N}_2}(a_1, \ldots, a_n) \coloneqq H_1\Big(g^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(b_n)\right)\Big).$$
- For $P \in \mathcal{L}$, an n-ary predicate symbol, and $a_1, \ldots, a_n \in N_2$, let
$$(a_1, \ldots, a_n) \in P^{\mathcal{N}_2} :\Longleftrightarrow \left(H_1^{-1}(b_1), \ldots, H_1^{-1}(b_n)\right) \in P^{\mathcal{N}_1}.$$

This way, $\mathcal{N}_2$ is defined, H_1 is an isomorphism and by Lemma 3.20, $\mathcal{N}_2$ is homogeneous. Moreover, for $a \in N_2$:

$$\begin{aligned} a \in U^{\mathcal{N}_2} &\overset{\text{def.}}{\Longleftrightarrow} H_1^{-1}(a) \in U^{\mathcal{N}_1} \\ &\overset{U^{\mathcal{N}_1} = U^{\mathcal{N}_0}}{\Longleftrightarrow} H_1^{-1}(a) \in U^{\mathcal{N}_0} \\ &\overset{\text{def. } H_1}{\Longleftrightarrow} H_0^{-1}(a) \in U^{\mathcal{N}_0} \\ &\overset{H_0 \text{ isom.}}{\Longleftrightarrow} a \in U^{\mathcal{N}_1}. \end{aligned}$$

In other words, $U^{\mathcal{N}_2} = U^{\mathcal{N}_1} = U^{\mathcal{N}_0}$. The last thing to check is that $\mathcal{N}_1 \prec \mathcal{N}_2$. First, recall, as we have already seen e.g. in Proposition 3.2, that whenever we have two $\mathcal{L}$-structures $\mathcal{A}, \mathcal{B}$ with an isomorphism $H : \mathcal{A} \cong \mathcal{B}$, then the following fact holds:

Claim: For all $t(x_1, \ldots, x_n) \in \mathrm{Tm}_\mathcal{L}$ and $a_1, \ldots, a_n \in B$, then

$$H\left(t^\mathcal{A}(H^{-1}(a_1), \ldots, H^{-1}(a_n))\right) = t^\mathcal{B}(a_1, \ldots, a_n).$$

Now, take $\phi(x_1, \ldots, x_n) \in \mathrm{Fml}_\mathcal{L}$ and $a_1, \ldots, a_n \in N_1$. We will prove that $\mathcal{N}_1 \prec \mathcal{N}_2$ by induction on the complexity of ϕ.

- If ϕ is $t_1 = t_2$:

$$\begin{aligned}
&\mathcal{N}_1 \vDash \phi(a_1, \ldots, a_n) \\
&\quad\iff t_1^{\mathcal{N}_1}(a_1, \ldots, a_n) = t_2^{\mathcal{N}_1}(a_1, \ldots, a_n) \\
&\quad\overset{H_0 \text{ isom.}}{\iff} H_0^{-1}\left(t_1^{\mathcal{N}_1}(a_1, \ldots, a_n)\right) = H_0^{-1}\left(t_2^{\mathcal{N}_1}(a_1, \ldots, a_n)\right) \\
&\quad\overset{\text{Claim for } H_0}{\iff} t_1^{\mathcal{N}_0}\left(H_0^{-1}(a_1), \ldots, H_0^{-1}(a_n)\right) = t_2^{\mathcal{N}_0}\left(H_0^{-1}(a_1), \ldots, H_0^{-1}(a_n)\right) \\
&\quad\overset{\mathcal{N}_0 \prec \mathcal{N}_1}{\iff} t_1^{\mathcal{N}_1}\left(H_0^{-1}(a_1), \ldots, H_0^{-1}(a_n)\right) = t_2^{\mathcal{N}_1}\left(H_0^{-1}(a_1), \ldots, H_0^{-1}(a_n)\right) \\
&\quad\overset{\text{def. } H_1}{\iff} t_1^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right) = t_2^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right) \\
&\quad\overset{H_1 \text{ isom.}}{\iff} H_1\left(t_1^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right)\right) = H_1\left(t_2^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right)\right) \\
&\quad\overset{\text{Claim for } H_1}{\iff} t_1^{\mathcal{N}_2}(a_1, \ldots, a_n) = t_2^{\mathcal{N}_2}(a_1, \ldots, a_n) \\
&\quad\iff \mathcal{N}_2 \vDash \phi(a_1, \ldots, a_n)
\end{aligned}$$

- If ϕ is $P(t_1, \ldots, t_m)$:

$$\begin{aligned}
&\mathcal{N}_1 \vDash \phi(a_1, \ldots, a_n) \\
&\quad\iff \left(t_1^{\mathcal{N}_1}(a_1, \ldots, a_n), \ldots, t_m^{\mathcal{N}_1}(a_1, \ldots, a_n)\right) \in P^{\mathcal{N}_1} \\
&\quad\overset{H_0 \text{ isom.}}{\iff} \left(H_0^{-1}\left(t_1^{\mathcal{N}_1}(a_1, \ldots, a_n)\right), \ldots, H_0^{-1}\left(t_m^{\mathcal{N}_1}(a_1, \ldots, a_n)\right)\right) \in P^{\mathcal{N}_0} \\
&\quad\overset{\text{Claim for } H_0}{\iff} \left(t_1^{\mathcal{N}_0}\left(H_0^{-1}(a_1), \ldots, H_0^{-1}(a_n)\right), \ldots, t_m^{\mathcal{N}_0}\left(H_0^{-1}(a_1), \ldots, H_0^{-1}(a_n)\right)\right) \in P^{\mathcal{N}_0} \\
&\quad\overset{\mathcal{N}_0 \prec \mathcal{N}_1}{\iff} \left(t_1^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right), \ldots, t_m^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right)\right) \in P^{\mathcal{N}_1} \\
&\quad\overset{H_1 \text{ isom.}}{\iff} \left(H_1\left(t_1^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right)\right), \ldots, H_1\left(t_m^{\mathcal{N}_1}\left(H_1^{-1}(a_1), \ldots, H_1^{-1}(a_n)\right)\right)\right) \in P^{\mathcal{N}_2} \\
&\quad\overset{\text{Claim for } H_1}{\iff} \left(t_1^{\mathcal{N}_2}(a_1, \ldots, a_n), \ldots, t_m^{\mathcal{N}_2}(a_1, \ldots, a_n)\right) \in P^{\mathcal{N}_2} \\
&\quad\iff \mathcal{N}_2 \vDash \phi(a_1, \ldots, a_n).
\end{aligned}$$

- If ϕ is $\neg\psi$:

$$\begin{aligned}\mathcal{N}_1 \vDash \phi(a_1,\dots,a_n) &\iff \mathcal{N}_1 \nvDash \psi(a_1,\dots,a_n)\\ &\overset{\text{I.H.}}{\iff} \mathcal{N}_2 \nvDash \psi(a_1,\dots,a_n)\\ &\iff \mathcal{N}_2 \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

- If ϕ is $\psi \wedge \theta$:

$$\begin{aligned}\mathcal{N}_1 \vDash \phi(a_1,\dots,a_n) &\iff \mathcal{N}_1 \vDash \psi(a_1,\dots,a_n) \text{ and } \mathcal{N}_1 \vDash \theta(a_1,\dots,a_n)\\ &\overset{\text{I.H.}}{\iff} \mathcal{N}_2 \vDash \psi(a_1,\dots,a_n) \text{ and } \mathcal{N}_2 \vDash \theta(a_1,\dots,a_n)\\ &\iff \mathcal{N}_2 \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

- If ϕ is $\exists x\psi(x,x_1,\dots,x_n)$:

$$\begin{aligned}\mathcal{N}_1 \vDash \phi(a_1,\dots,a_n) &\implies \text{there is some } a \in N_1 : \mathcal{N}_1 \vDash \psi(a,a_1,\dots,a_n)\\ &\overset{\text{I.H}}{\implies} \text{there is some } a \in N_1 \subseteq N_2 : \mathcal{N}_2 \vDash \psi(a,a_1,\dots,a_n)\\ &\implies \mathcal{N}_2 \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

  Conversely, suppose that $\mathcal{N}_2 \vDash \phi(a_1,\dots,a_n)$. Thus, there exists some $a \in N_2$ such that $\mathcal{N}_2 \vDash \psi(a,a_1,\dots,a_n)$. Let $b \coloneqq H_1^{-1}(a) \in N_1$ and $b_i \coloneqq H_1^{-1}(a_i) \in N_1$ for $1 \leqslant i \leqslant n$. Then, recalling that by Proposition 3.2 isomorphisms are in particular elementary embeddings, we have:

$$\begin{aligned}\mathcal{N}_2 \vDash \psi(H_1(b),H_1(b_1),\dots,H_1(b_n)) &\overset{H_1 \text{ elem. embedd.}}{\implies} \mathcal{N}_1 \vDash \psi(b,b_1,\dots,b_n)\\ &\overset{\text{I.H}}{\implies} \mathcal{N}_2 \vDash \psi(b,b_1,\dots,b_n)\\ &\implies \mathcal{N}_2 \vDash \phi(b_1,\dots,b_n)\\ &\overset{H_1^{-1} \text{ elem. embedd.}}{\implies} \mathcal{N}_1 \vDash \phi(a_1,\dots,a_n).\end{aligned}$$

If $\alpha < \omega_1$ is a limit ordinal, let

$$\mathcal{N}_\alpha \coloneqq \bigcup_{\beta<\alpha} \mathcal{N}_\beta.$$

Observe that for all $\beta < \alpha$:

- $\mathcal{N}_\beta \prec \mathcal{N}_\alpha$ by the theorem of union of chains.
- $N_\beta \subsetneq N_{\beta+1} \subseteq N_\alpha$.
- $\mathcal{N}_\alpha$ is homogeneous by Theorem 3.21
- Take $C \subseteq N_\alpha$ finite. We know that $C \subseteq N_\gamma$ for some $\gamma < \alpha$. Hence $\mathcal{N}_\alpha|_C = N_\gamma|_C$, which is isomorphic to a finitely generated substructure of $\mathcal{N}_0$, since $\mathcal{N}_0 \cong \mathcal{N}_\gamma$. Therefore, $\mathcal{N}_\alpha \cong \mathcal{N}_0$ by Theorem 3.23 and thus $\mathcal{N}_\alpha \cong \mathcal{N}_\beta$.

- $U^{\mathcal{N}_\alpha} = \bigcup_{\gamma<\alpha} U^{\mathcal{N}_\gamma} = \bigcup_{\gamma<\alpha} U^{\mathcal{N}_0} = U^{\mathcal{N}_0}$.

The only case left, is when α is a successor but not a double successor ordinal. For that, let us look at the construction of $\mathcal{N}_{\omega+1}$. We can simply repeat the argument done in the double successor case, by taking a bijective copy X_ω of $N_\omega \setminus N_0$, letting $N_{\omega+1} \coloneqq N_\omega \sqcup X_\omega$ and constructing an isomorphism H_ω which extends $\mathcal{N}_0 \cong \mathcal{N}_\omega$ as on the diagram.

$$\begin{array}{ccc} N_0 & \xrightarrow{\quad\cong\quad} & N_\omega \\ \vdots & & \vdots \\ N_\omega = N_0 \sqcup (N_\omega \setminus N_0) & \overset{H_\omega}{\dashrightarrow} & N_{\omega+1} = N_\omega \sqcup X_\omega \\ \vdots & & \vdots \\ N_\omega \setminus N_0 & \xrightarrow{\quad\approx\quad} & X_\omega \end{array}$$

We finally end up with an elementary chain $\langle \mathcal{N}_\alpha : \alpha < \omega_1 \rangle$. We now define

$$\mathcal{N}_{\omega_1} \coloneqq \bigcup_{\alpha<\omega_1} \mathcal{N}_{\omega_1}.$$

Observe that

- $U^{\mathcal{N}_{\omega_1}} = \bigcup_{\alpha<\omega_1} U^{\mathcal{N}_\alpha} = \bigcup_{\alpha<\omega_1} U^{\mathcal{N}_0} = U^{\mathcal{N}_0}$, which has cardinality $\aleph_0$.
- N_{ω_1} is of cardinality $\aleph_1$.
- By the theorem of union of chains, $\mathcal{N}_1 \prec \mathcal{N}_{\omega_1}$, and since $\mathcal{N}_1 \vDash T$, we also have $\mathcal{N}_{\omega_1} \vDash T$.

We have proved that T admits $\langle \aleph_1, \aleph_0 \rangle$, which concludes the proof. □

Vaught theorem is then the immediate consequence of Lemma 3.46 and Lemma 3.47.

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