Blakers-Massey Connectivity Theorem from the perspective of homotopy type theory

Master thesis of Aloïs Rosset

Supervised by Jérôme Scherer

Abstract

The famous result of homotopy theory known as the Blakers-Massey Connectivity Theorem is investigated here from the viewpoint of homotopy type theory.

Type theory

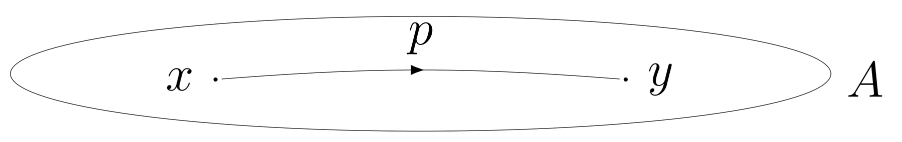
Type theory is an alternative approach to set theory as a *foundation of mathematics*. It presents some advantages, with for instance its applications in computer sciences and the very structured logical formalism which is employed. The basic concept is to have a *term* a of a *type* A, denoted $a : A$. It is analogous to the set theoretic affirmation "a is an element of the set A", or even more to the homotopic idea that "a is a point of the space A". It has the specificity of being a minimalist approach, where objects are reduced to their core aspects.

Among the primary ways of combining types to create new ones, the most characteristic ones are the dependent functions $\Pi_{(x:A)}B(x)$ and the dependent pairs $\Sigma_{(x:A)}B(x)$.

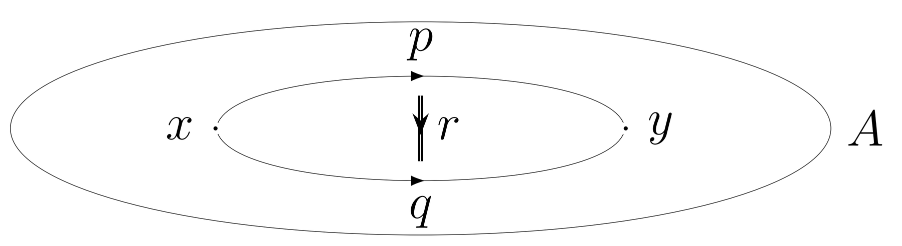
In particular thanks to those two symbols, type theory contains the beautiful notion of propositions as types where mathematical statements can be formulated as types and then proved by exhibiting an inhabitant of the concerned type.

Homotopy type theory

The revolutionary idea of *homotopy type theory* is to make use of the intrinsic homotopic content of type theory. For that, any witness of an equality $p : x =_A y$ is viewed as a *path* in the space A.



Following this idea, if $p, q : x =_A y$ are two paths, witnessing their equality $r : p = q$ comes down to a *homotopy of paths* or *2-path*.

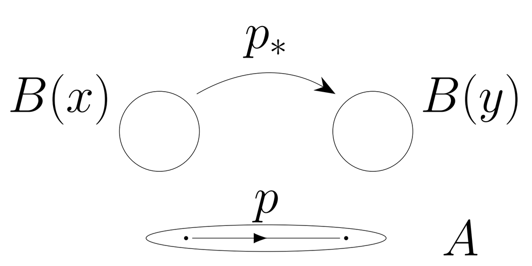


Iterating with equalities then iterates also on the dimension with *3-paths*, *4-paths*, etc. The equality-path type behaves then as expected, with constant paths, paths inversion and paths concatenation, with the habitual properties being carried over.

Those properties includes notably the observation that functions are functorial, as seen by the fact that they preserve equalities, i.e., given a function f, then

$$(x = y) \to (fx = fy)$$
$$p \mapsto fp.$$

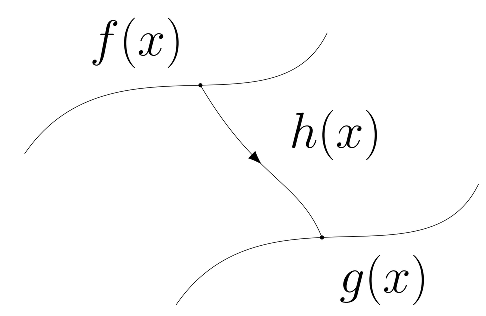
Another core observation is that a family of types $B : A \to U$ induces a fibration. Each $B(x)$ works as a fiber over $x : A$, with the total space $\Sigma_{(x:A)}B(x)$ being all those fibers put together and indexed by A. Then, whenever two points are linked $p : x = y$, there is a transport functions between fibers.



Equivalence

The equality gives homotopies of paths, but there is also a type for homotopies between two functions $f, g : \Pi_{(x:A)}B(x)$,

$$f \sim g :\equiv \Pi_{(x:A)}(f(x) =_{B(x)} g(x)).$$



There are many ways to define a notion of equivalence in HoTT, for instance by taking the literal translation of homotopy equivalence. In the end, we can express when a function f is an equivalence by a type $\mathsf{isequiv}(f)$, and when types are equivalent with

$$(A \simeq B) :\equiv \Sigma_{(f:A\to B)}\mathsf{isequiv}(f).$$

Axioms

The *function extensionality axiom* states that functions f, g are equal if and only if they are equal at each of their values:

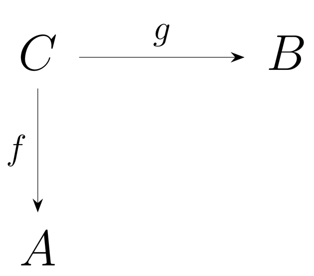
$$(f = g) \simeq (f \sim g),$$

In fact, it derives from the real major innovation of HoTT, the *univalence axiom*, which states that isomorphic types as treated as equal and vice-versa:

$$(A \simeq B) \simeq (A = B).$$

Pushout

Given a span

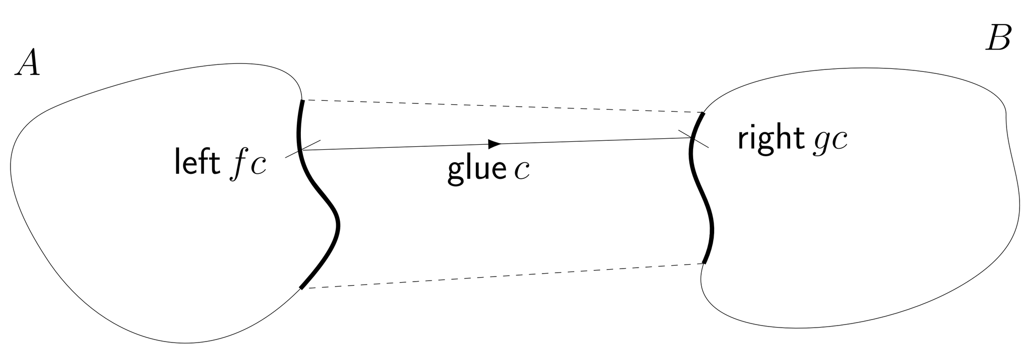


the *(homotopy) pushout* $A \sqcup_C B$ is generated by two functions that inject both A and B,

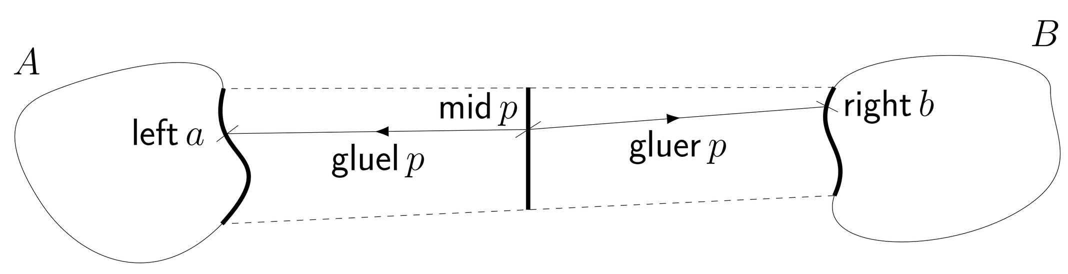
- $\mathsf{left} : A \to A \sqcup_C B$ and
- $\mathsf{right} : B \to A \sqcup_C B$,

and by a third function gluing corresponding points

- $\mathsf{glue} : \Pi_{(c:C)}\,\mathsf{left}\,fc = \mathsf{right}\,gc$.



The type C can equivalently be swapped with a predicate $P : A \to B \to \mathcal{U}$. With this latter, the pushout admits also a construction with two "gluing cylinders".



Connectivity

A type A is an *n-type*, for $n \geqslant -2$, if informally A has a trivial homotopic structure in degree $n+1$ and above, which is easy to formulate thanks to the innate structure of ∞-groupoid inside types.

Then, an arbitrary type B can be transformed into an n-type by the construction of its *n-truncation* $\|B\|_n$.

Dually, a type C is *n-connected*, for $n \geqslant -2$, if it has trivial homotopic structure in degree n and below, which can be stated as $\|C\|_n$ being contractible.

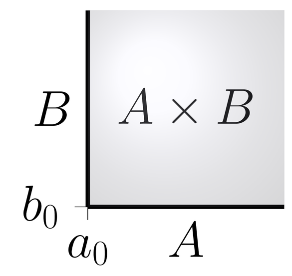
Then, a function $f : D \to E$ is called n-connected if all its homotopy fibers are n-connected, where the *(homotopy) fiber* over a point $e : E$ is

$$\mathsf{fib}_f(e) :\equiv \Sigma_{(d:D)}(f(d) = e).$$

Wedge product

A common construction, given pointed types (A, a_0) and (B, b_0), is the pushout of $A \xleftarrow{a_0} \mathbf{1} \xrightarrow{b_0} B$, called the *wedge* $A \vee B$. The idea is to obtain the union of A and B which is almost disjoint, with only the two points a_0 and b_0 being glued together. A natural map exists:

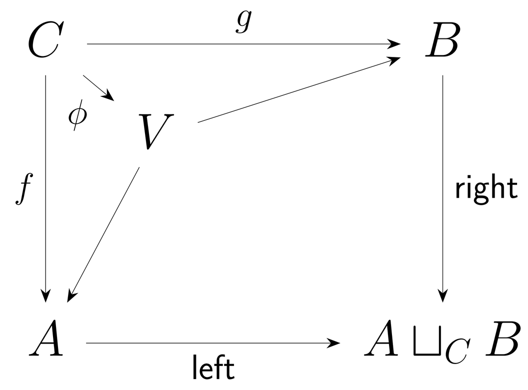
$$\mathsf{wtp} : A \vee B \to A \times B,$$



A central prerequisite for the Blakers-Massey theorem is the Wedge Connectivity Lemma, which states that if (A, a_0) and (B, b_0) are respectively i- and j-connected, then wtp is $(i+j)$-connected.

Blakers-Massey theorem

Take a pushout and a pullback as in the diagram,



where $V :\equiv A \times_{A \sqcup_C B} B$ and ϕ is the natural function. If f and g are i- and j-connected respectively, then ϕ is $(i+j)$-connected.

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