Quasitoposes in graph rewriting

Fuzzy presheaves and LT-topologies

Aloïs Rosset

02 Oct 2025, **EPFL Topology Seminar**

VRIJE UNIVERSITEIT AMSTERDAM

Rewriting

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Comparison

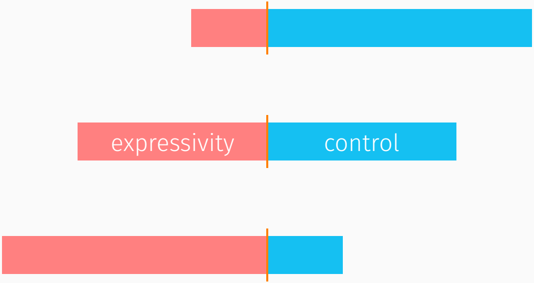
String rewriting:

$\subseteq$

Term rewriting:

$\subseteq$

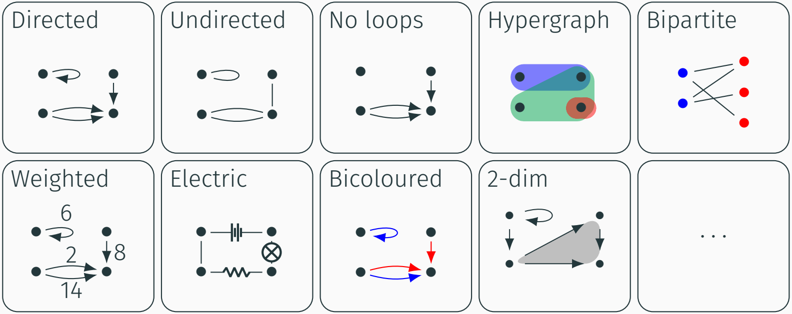
Graph rewriting:



Challenge: lift algorithms and results: string $\rightsquigarrow$ terms $\rightsquigarrow$ graphs.

Graph rewriting

Which graphs?



$\Longrightarrow$ Categorical framework

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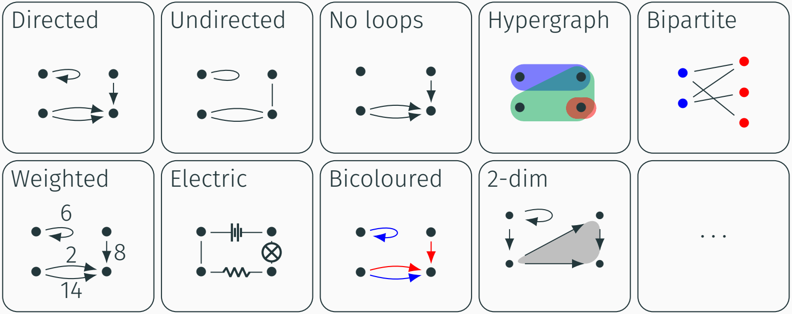


$\Longrightarrow$ Categorical framework

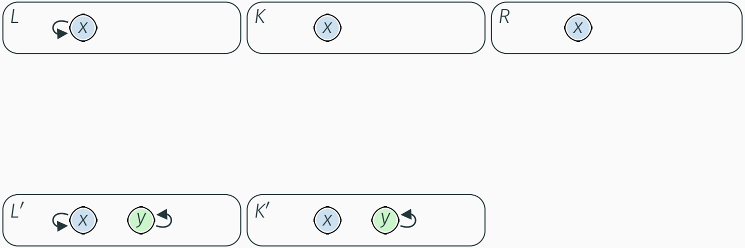
Categorical formalisms/algorithms use **pushouts** and **pullbacks**.

Recent work: Overbeek, Endrullis, Rosset (ICGT 21, JLAMP'23)

Pullback-Pushout Plus (PBPO$^+$)

$$\begin{tikzcd} L \arrow[d, tail] & K \arrow[l] \arrow[r] \arrow[d, tail] \arrow[dl, phantom, "\text{PB}"] & R \\ L' & K' \arrow[l] & \end{tikzcd}$$

rewrite rule



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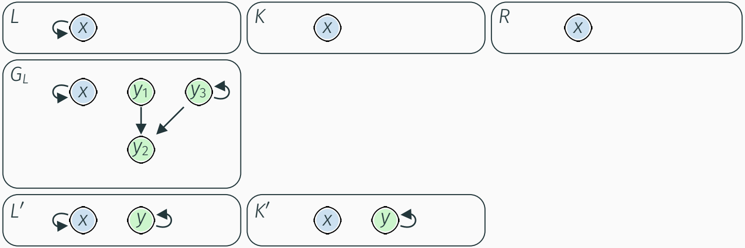

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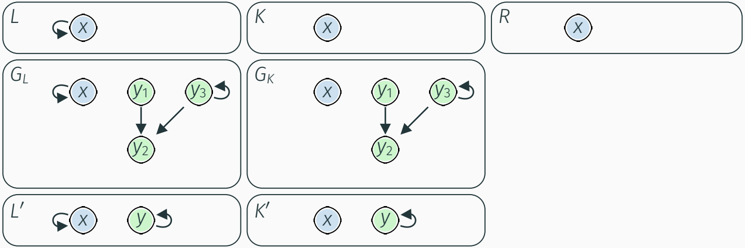

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G_L \arrow[d] & G_K \arrow[l] \arrow[r] \arrow[d] \arrow[dl, phantom, "\mathbf{PB}"] & G_R \\
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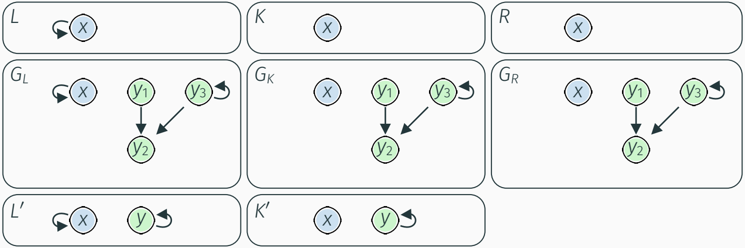

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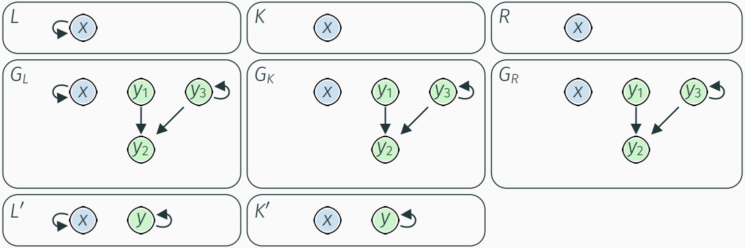
rewrite rule

rewrite step $G_L \to G_R$



Advantages of PBPO$^+$: Relabelling

Label vertices and edges, e.g., with $\mathcal{L} = \{a, b, c\}$.

Suppose complete lattice structure $(\mathcal{L}, \vee, \wedge, \top, \perp)$, e.g., $\perp \begin{matrix} \leq \\ \leq \\ \leq \end{matrix} \begin{matrix} a \\ b \\ c \end{matrix} \begin{matrix} \geq \\ \geq \\ \geq \end{matrix} \top$

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Definition

Fuzzy set (A, α) is a set A with a membership function $\alpha : A \to \mathcal{L}$.

Fuzzy graph (A, α) is a graph A with $\begin{cases} \alpha_E : A(E) \to \mathcal{L} \\ \alpha_V : A(V) \to \mathcal{L} \end{cases}$

Morphisms $f : (A, \alpha) \to (B, \beta)$ must not decrease membership: $\alpha \leqslant \beta \circ f$.

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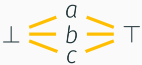

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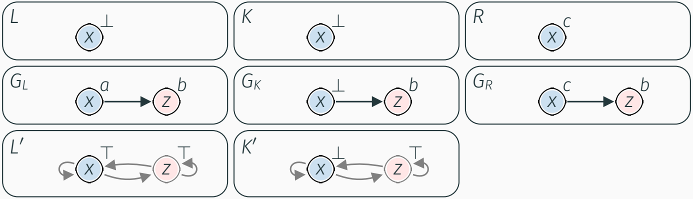

Toposes

Motivations for toposes

Theorem (Overbeek, Endrullis, Rosset (ICGT'21, JLAMP'23))

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Theorem (Behr, Harmer, Krivine (ICGT'21))

- *In quasitoposes: concurrency property in non-linear SqPO.*
- *In rm-adhesive categories: concurrency property in non-linear DPO.*

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$$\begin{tikzcd}
\text{Grothendieck topos} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Elementary topos} \arrow[d, Rightarrow] \\
\text{Grothendieck quasitopos} \arrow[r, Rightarrow] & \text{Elementary quasitopos}
\end{tikzcd}$$

Example of toposes: presheaf categories.

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Presheaf morphism $f : A \Rightarrow B$ is a natural transformation.

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$\mathrm{sym} \circlearrowright E \overset{s}{\underset{t}{\rightrightarrows}} V$ $\mathrm{sym} \cdot \mathrm{sym} = \mathrm{id}_E,$ $s \cdot \mathrm{sym} = t,$ $t \cdot \mathrm{sym} = s.$	Undirected graphs

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$E \underset{t}{\overset{s}{\rightrightarrows}} V$	Graph (directed multi-)	$\mathrm{sym} \circlearrowleft E \underset{t}{\overset{s}{\rightrightarrows}} V$ $\mathrm{sym} \cdot \mathrm{sym} = \mathrm{id}_E$, $s \cdot \mathrm{sym} = t$, $t \cdot \mathrm{sym} = s$.	Undirected graphs
$E \;\substack{\xrightarrow{s_1} \\ \vdots \\ \xrightarrow{s_k}}\; V$	directed k-uniform hypergraphs	$E \;\substack{\xrightarrow{s} \\ \xleftarrow{\mathrm{refl}} \\ \xrightarrow{t}}\; V$ $s \cdot \mathrm{refl} = t \cdot \mathrm{refl} = \mathrm{id}_V$	Reflexive graphs (or *degenerate* graphs)

Yoneda embedding

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Idea: It gives the “building blocks” of presheaf categories.

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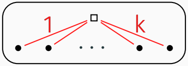
$E \begin{array}{c} \xrightarrow{s_1} \\ \vdots \\ \xrightarrow{s_k} \end{array} V$	*k*-uniform hypergraphs	$\bullet$	

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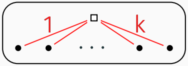
$E \begin{smallmatrix} \xrightarrow{s_1} \\ \cdots \\ \xrightarrow{s_k} \end{smallmatrix} V$	k-uniform hypergraphs	$\bullet$	
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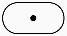
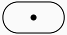
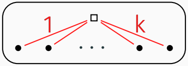
$E \overset{s}{\underset{t}{\rightrightarrows}} V$	Graph		
$E \begin{array}{c}\xrightarrow{s_1}\\ \cdots \\ \xrightarrow{s_k}\end{array} V$	k-uniform hypergraphs		
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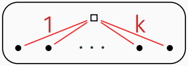
$E \begin{array}{c}\xrightarrow{s_1}\\ \cdots \\ \xrightarrow{s_k}\end{array} V$	k-uniform hypergraphs	$\bullet$	
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Given category I and functor $\mathcal{L} : I^{\mathrm{op}} \to \mathrm{Poset}$:

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I^{op}	i	$E \overset{s}{\underset{t}{\rightrightarrows}} V$	$E \;\substack{\xrightarrow{s_1} \\ \xrightarrow{s_2} \\ \xrightarrow{s_3}}\; V$	...
Fuzzy presheaf category	Fuzzy sets	Fuzzy graphs	Fuzzy 3-uniform hypergraphs	...

Presheaves properties

Let I be a small category.

$$\begin{array}{rcl} \mathsf{Set} & \Longrightarrow & \mathsf{Topos} \\ \mathsf{Graph} & \Longrightarrow & \mathsf{Topos} \\ & \dots & \\ \mathsf{Set}^{I^{\mathrm{op}}} & \Longrightarrow & \mathsf{Topos} \end{array} \qquad \begin{array}{rcl} \mathsf{FuzzySet}(\mathcal{L}) & \overset{(\star)\text{Stout'93}}{\Longrightarrow} & \mathsf{Quasitopos} \\ & & \\ & \dots & \\ & & \end{array}$$

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Bonus: FuzzyPresheaf($I, \mathcal{L}$) is rm-adhesive.

Proof

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Terminal object + Pullbacks $\overset{\text{e.g., Borceux Vol1}}{\Longrightarrow}$ all finite limits

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An object $\mathbf{1}$ is **terminal** if $\forall$ object A, $\exists!$ arrow $A \to \mathbf{1}$.

Definition (Product of objects A, B)

Product is object $A \times B$ with $\pi_1 : A \times B \to A$, and $\pi_2 : A \times B \to B$, + a universal property.

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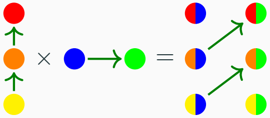
Graphs	$\{\ \cdot \circlearrowleft\}$	

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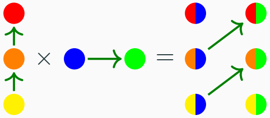
Graphs	$\{\ \cdot \circlearrowleft\}$	
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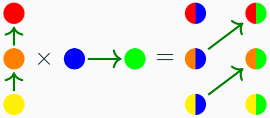
Graphs	$\{\ \cdot\ \circlearrowleft\}$	
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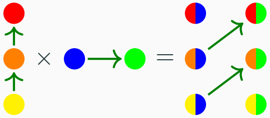
Graphs	$\{\ \cdot \circlearrowleft\}$	
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Subobject classifier

Definition

Subobject classifier is True : $1 \to \Omega$ with

$$\underset{\text{subobjects}}{A \subseteq B} \quad \Longleftrightarrow \quad \underset{\text{characteristic function}}{\chi_A : B \to \Omega}$$

$$\begin{tikzcd} A \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\text{True}"] \\ B \arrow[r, "\chi_A"'] & \Omega \end{tikzcd}$$

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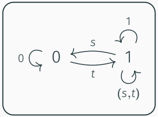
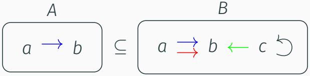
- Graph : $\Omega \coloneqq$ 



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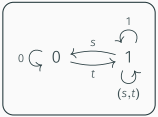
- Graph : $\Omega \coloneqq$ 

$$\overset{A}{\boxed{a \to b}} \subseteq \overset{B}{\boxed{a \rightrightarrows b \leftarrow c \circlearrowleft}} \iff \chi_A : \begin{cases} a, b & \mapsto 1 \\ c & \mapsto 0 \\ \to & \mapsto \xrightarrow{1} \\ \to & \mapsto \xrightarrow{(s,t)} \\ \leftarrow & \mapsto \xrightarrow{t} \\ \circlearrowleft & \mapsto \xrightarrow{0} \end{cases}$$

- Presheaves: $\Omega(i) = \mathrm{Sub}(y(i))$ is the set of all subpresheaves/sieves of $y(i)$. (generalisation of sets and graphs)

Regular-subobject classifier

- FuzzySet($\mathcal{L}$): ✗ **However,** $\Omega := \{0^\top, 1^\top\}$ classifies regular fuzzy subsets.

$$\{a^x\} \subseteq \{a^{0.4}, b^{0.6}\} \text{ is } \textbf{regular} \quad \Longleftrightarrow \quad x = 0.4$$

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Lemma

Regular-subobject classifier fuzzy presheaves:

$$\Omega \textit{ of presheaves} + \textit{all elements full membership.}$$

Cartesian closed

Definition

For objects A, B of a category, an **exponential object** is an object B^A s.t.

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Lemma

Fuzzy presheaves have exponential objects.

Locally cartesian closed

Cartesian closedness stable under categorical equivalence. Hence:

Lemma (e.g., Awodey'05)

Presheaf(I) *is locally cartesian closed because*

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Lemma

FuzzyPresheaf($I, \mathcal{L}$) *is locally cartesian closed because*

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Adhesivity

Definition (Simplified)

Category is

- **adhesive**: if m mono, bottom PO, back PBs:

  front PBs $\iff$ top PO
- **rm-adhesive**: if m reg. mono, ——〃——:

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- **rm-quasiadhesive**: if ——〃——:

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$$\begin{tikzcd}
 & F \arrow[rr] \arrow[dl] \arrow[dd] & & G \arrow[dl] \arrow[dd] \\
E \arrow[rr, crossing over] \arrow[dd] & & H & \\
 & B \arrow[rr] \arrow[dl, "m"] & & C \arrow[dl] \\
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Theorem

$$
\begin{tikzcd}
\mathit{Topos} \arrow[rr, Rightarrow, "\text{Lack, Sobociński '06}"] \arrow[dd, Rightarrow] & & \mathit{Adhesive} \arrow[d, Rightarrow] \\
 & & Rm-adhesive \arrow[d, Rightarrow] \\
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 & & & Rm-adhesive \arrow[d, Rightarrow] \\
Quasitopos \arrow[urrr, Rightarrow, "\text{if regular subobjects closed under binary union Johnstone et al. '07}"] \arrow[rrr, Rightarrow, "\text{Garner, Lack '12}"'] & & & Rm-quasiadhesive
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LT-Topologies

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Toposes have an internal logic (Goldblatt'06, McLarty'92)

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Graph : $_0 \circlearrowleft 0$, $_{(s,t)} \circlearrowleft 1$, $_1 \circlearrowleft 1$

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Logical connectives:

$$
\begin{aligned}
\mathsf{False} &\coloneqq \chi_{0 \overset{!}{\rightarrowtail} 1} \\
\neg &\coloneqq \chi_{\mathsf{False}} \\
\wedge &\coloneqq \chi_{\langle \mathsf{True}, \mathsf{True} \rangle} \\
\Rightarrow &\coloneqq \chi_{\mathsf{Eq}(\pi_1, \wedge) \to \Omega \times \Omega} \\
\vee &\coloneqq \chi_{[\langle \mathsf{True}_\Omega, \mathrm{id}\Omega \rangle, \langle \mathrm{id}\Omega, \mathsf{True}_\Omega \rangle]}
\end{aligned}
$$

Logic with Ω

Toposes have an internal logic (Goldblatt'06, McLarty'92)

Definition

Truth values are $1 \to \Omega$ (global elements).

Set : 0, 1 Graph : $0 \circlearrowleft 0$, $(s,t) \circlearrowleft 1$, $1 \circlearrowleft 1$

Logical connectives:

$$\text{False} \coloneqq \chi_{0 \overset{!}{\rightarrowtail} 1}$$

$$\neg \coloneqq \chi_{\text{False}}$$

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$$\vee \coloneqq \chi_{[\langle \text{True}_\Omega, \mathrm{id}\Omega \rangle, \langle \mathrm{id}\Omega, \text{True}_\Omega \rangle]}$$

$\implies$ Ω is a Heyting algebra.

(for object X, then $\text{True}_X \coloneqq X \xrightarrow{!} 1 \xrightarrow{\text{True}} \Omega$ means true *in context* X)

Lawvere-Tierney Topologies

LT-topology / operator / modality talks about “locally truth”.

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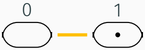
Induces ***closure*** operator on subobjects: $\overline{A_0 \rightarrowtail A}$: $\chi_{\overline{A_0 \rightarrowtail A}} := j \circ \chi_{A_0 \rightarrowtail A}$.

Subobject $A_0 \rightarrowtail A$ is **dense** if $\overline{A_0} = A$.

Examples of topologies

	Topology j	Closure $\overline{A_0 \subseteq A}$	Dense object
Set	*Trivial* True_Ω	adds everything	all
	Discrete $\mathrm{id}\Omega$	adds nothing	only A

$y(V) =$ 

$\Omega(V) = \mathrm{Sub}(y(V)) =$ 

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Graph	*Dbl. negation* $\neg\neg$	adds all valid edges	if $A_0(V) = A(V)$

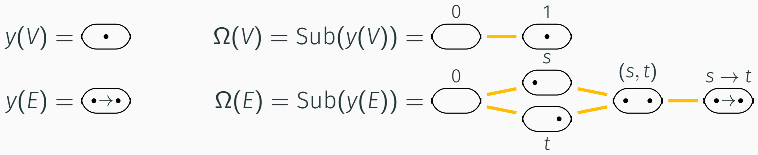

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Illustration of **$\neg\neg$-topology** on Graph: for A: $\boxed{a \rightrightarrows b \leftarrow c \circlearrowleft}$

A_0: $\boxed{a \rightarrow b}$ $\qquad$ $\neg A_0$: $\boxed{c \circlearrowleft}$ $\qquad$ $\neg\neg A_0$: $\boxed{a \rightrightarrows b}$

Separated elements

Hausdorff space B: any $A \xrightarrow{f} B$ determined by values on any dense $A_0 \subseteq A$.

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Definition

Topology j, object B, arbitrary j-dense $m : A_0 \rightarrowtail A$ and arbitrary $f : A_0 \to B$.
Count the number of factorisations $g : A \to B$ of f through m $(\forall A_0, A, f)$:

- B is ***separated*** if $\#g \leqslant 1$,
- B is ***complete*** if $\#g \geqslant 1$,
- B is a ***sheaf*** if $\#g = 1$.

$$\begin{tikzcd} A_0 \arrow[d, tail, "m"'] \arrow[dr, "f"] & \\ A \arrow[r, dotted, "g"'] & B \end{tikzcd}$$

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Lemma (Johnstone '79)

For a topology,

- *separated elements form a quasitopos.*
- *sheaves form a quasitopos.*

Separated elements and sheaves in Set, Graph, FuzzyGraph($\mathcal{L}$)

Top.	Closure $\overline{A_0 \subseteq A}$	Dense object	Sep. elem.	Sheaves
Triv.	(adds everything)	all	subterminal objects	terminal objects
Dis.	(adds nothing)	only A	(all)	all
Closed	(adds all vertices)	if $A_0(E) = A(E)$	$\varnothing$, 	
$\neg\neg$	(adds all valid edges)	if $A_0(V) = A(V)$	simple graphs	complete graphs

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Corollary (Vigna'97)

Simple graphs form a quasitopos.

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Corollary (Vigna'97)

Simple graphs form a quasitopos.

Lemma

Similarly, those also form quasitoposes:

- *Simple reflexive graphs*
- *Simple undirected graphs*
- *Simple k-uniform hypergraphs*
- ...

Bicoloured

Bicoloured graphs

I^{op}	$\mathrm{Set}^{I^{\mathrm{op}}}$
$E \underset{t}{\overset{s}{\rightrightarrows}} V$	Graph
$\color{blue}{E} \underset{t}{\overset{s}{\rightrightarrows}} V \underset{t'}{\overset{s'}{\leftleftarrows}} \color{red}{E'}$	Bicoloured graphs ($\color{blue}{\rightarrow}$ and $\color{red}{\rightarrow}$)

Bicoloured graphs

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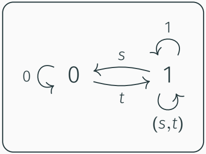
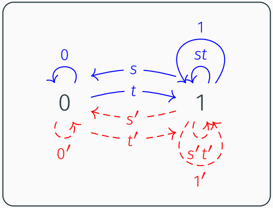
$E \overset{s}{\underset{t}{\rightrightarrows}} V$	Graph
$E \overset{s}{\underset{t}{\rightrightarrows}} V \overset{s'}{\underset{t'}{\leftleftarrows}} E'$	Bicoloured graphs ($\rightarrow$ and $\rightarrow$)

Ω_{Graph}



$\Omega_{\mathrm{BiColGraph}}$



Topologies on BiColGraph

Lemma

4 *topologies on* Graph $\implies$ 8 *topologies on* BiColGraph.

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	j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8
on E	*disc.*	$\neg\neg$	*disc.*	$\neg\neg$	*closed*	*closed*	*triv.*	*triv.*
on E'	*disc.*	*disc.*	$\neg\neg$	$\neg\neg$	*closed*	*triv.*	*closed*	*triv.*

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on E	disc.	$\neg\neg$	disc.	$\neg\neg$	closed	closed	triv.	triv.
on E'	disc.	disc.	$\neg\neg$	$\neg\neg$	closed	triv.	closed	triv.

$$\left.\begin{array}{l} j_2\text{-separated el.} = \text{blue-simple graphs.} \\ j_3\text{-separated el.} = \text{red-simple graphs.} \end{array}\right\} \implies \text{Partially simple graphs.}$$

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Corollary

Partially simple graphs form a quasitopos.

Simplicial sets

Simplicial sets

Simplicial sets = graphs with triangles (2-dim), tetrahedra (3-dim), etc.

Definition

sSet = **Simplicial sets** $= \mathrm{Set}^{\Delta^{\mathrm{op}}}$ with

$$
\Delta^{\mathrm{op}} = \quad
\begin{tikzcd}
0 & 1 \arrow[l, shift left=4, "d_1^1"'] \arrow[l, shift right=4, "d_0^1"] & 2 \arrow[l, shift left=6, "d_2^2"'] \arrow[l, "d_1^2"'] \arrow[l, shift right=6, "d_0^2"] & \cdots & \cdots
\end{tikzcd}
\qquad 0 \xrightarrow{s_0^0} 1, \quad 1 \xrightarrow{s_0^1} 2, \quad 1 \xrightarrow{s_1^1} 2
$$

d_i^n = **face maps**, s_i^n = **degeneracy maps** satisfy *simplicial identities*

sSet_+ = **Semi-simplicial sets**: only d_i maps.
$\mathrm{sSet}^{\leqslant n}$ = n**-dimensional simplicial sets**: only $0, \ldots, n$.
$\mathrm{sSet}_+^{\leqslant n}$ = n**-dimensional semi-simplicial sets**: both.

Simplicial identities

Δ is defined with totally ordered sets. Δ^{op} is defined with graphs.

Map d_i *omits* vertex at position i: $d_1(v_0, v_1) = v_0$.
Map s_i *duplicates* vertex at position i: $s_1(v_0, v_1) = (v_0, v_1, v_1)$.

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Simplicial identities:
$$\begin{cases} d_i^{n-1} d_j^n = d_{j-1}^{n-1} d_i^n & \text{if } i < j \\ s_i^{n+1} s_j^n = s_{j+1}^{n+1} s_i^n & \text{if } i \leqslant j \\ d_i^{n+1} s_j^n = \begin{cases} s_{j-1}^{n-1} d_i^n & \text{if } i < j \\ \mathrm{id}_n & \text{if } i = j \text{ or } i = j+1 \\ s_j^{n-1} d_{i-1}^n & \text{if } i > j+1 \end{cases} \end{cases}$$

$$\begin{tikzcd} & 1 \arrow[dd, "{(1,2)}"] \\ 0 \arrow[ur, "{(0,1)}"] \arrow[dr, "{(0,2)}"'] \arrow[r, phantom, "{(0,1,2)}"] & {} \\ & 2 \end{tikzcd}$$

$$\begin{array}{rclclclcl} d_1 d_2(0,1,2) & = & d_1(0,1) & = & 0 & = & d_1(0,2) & = & d_1 d_1(0,1,2) \\ s_0 s_0(0) & = & s_0(0,0) & = & (0,0,0) & = & s_1(0,0) & = & s_1 s_0(0) \\ d_0 s_1(1,2) & = & d_0(1,2,2) & = & (2,2) & = & s_0(2) & = & s_0 d_0(1,2) \end{array}$$

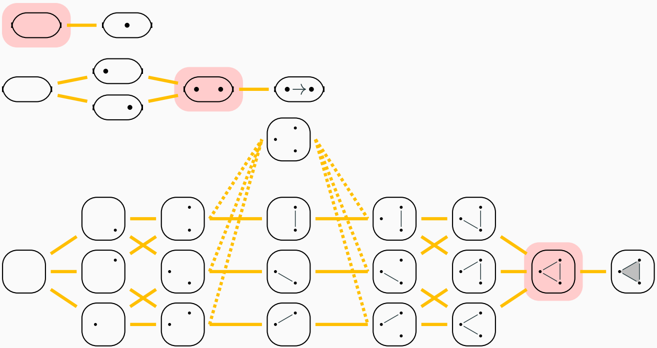
$\mathrm{sSet}_+^{\leqslant n}$: Intuition

Case $\mathrm{sSet}_+^{\leqslant n}$. **Inductively**: at each dimension, is the new structure added?

$\Omega(0) =$

$\Omega(1) =$

$\Omega(2) =$



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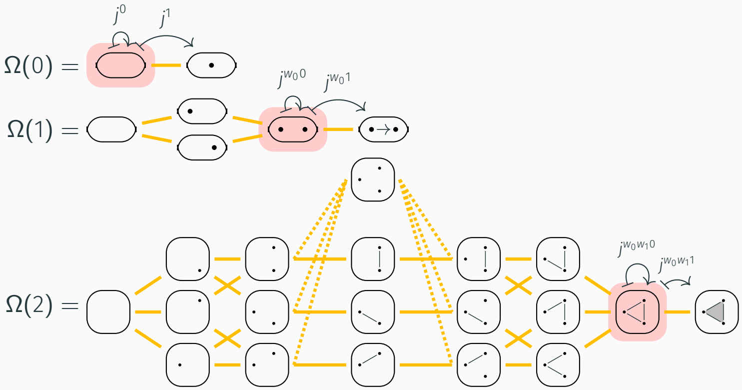

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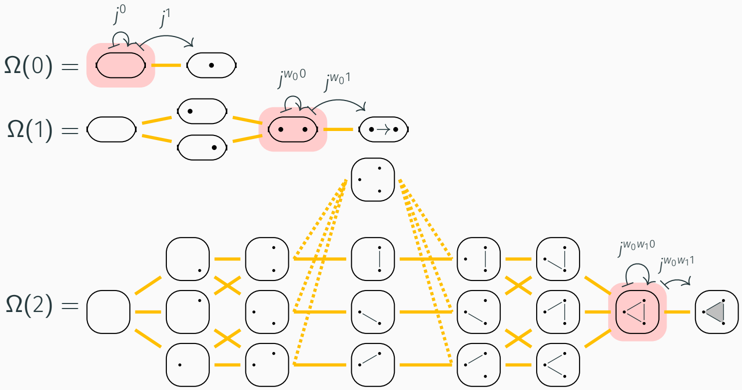


Theorem

Topologies on $\mathrm{sSet}_+^{\leqslant n}$: exactly the unique extensions of some j^w.

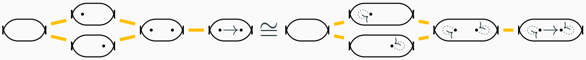
Discrete topology: $w = 0 \dots 0$. Trivial topology: $w = 1 \dots 1$.

$\mathrm{sSet}^{\leqslant n}$: adding degeneracies

Fact (easy proof requires folklore knowledge about a left Kan extension)

$$\Omega_+^{\leqslant n}(k) \cong \Omega^{\leqslant n}(k)$$

Example:

$\Omega_+^{\leqslant 1}(1) =$  $= \Omega^{\leqslant 1}(1)$

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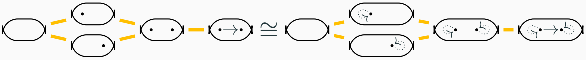
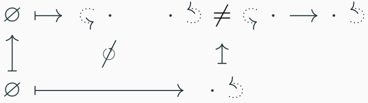
$\Omega_+^{\leqslant 1}(1) =$  $= \Omega^{\leqslant 1}(1)$

In $\mathrm{sSet}_+^{\leqslant 1} = \mathrm{Graph}$: 4 topologies $j^{00}, j^{01}, j^{10}, j^{11}$.
In $\mathrm{sSet}^{\leqslant 1} = \mathrm{ReflGraph}$: 3 topologies $j^{00}, j^{01}, \quad j^{11}$
Indeed, j^{10} does not commute with the degeneracy map $\Omega^{\leqslant 1}(\mathrm{refl})$:

$$\begin{tikzcd} \Omega^{\leqslant 1}(1) \arrow[r, "j_1^{10}"] & \Omega^{\leqslant 1}(1) \\ \Omega^{\leqslant 1}(0) \arrow[u, "\Omega^{\leqslant 1}(\mathrm{refl})"] \arrow[r, "j_0^{10}"'] & \Omega^{\leqslant 1}(0) \arrow[u, "\Omega^{\leqslant 1}(\mathrm{refl})"'] \end{tikzcd}$$



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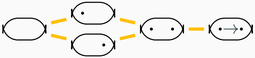
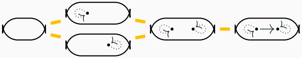
$\Omega_+^{\leqslant 1}(1) =$  $\cong$  $= \Omega^{\leqslant 1}(1)$

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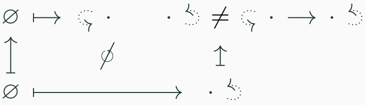

Theorem

Topologies on $\mathrm{sSet}^{\leqslant n}$ are of the form j^w for $w = 0^m 1^{n+1-m}$.

sSet_+ and sSet

General case = extend the *n*-dimensional case

Theorem

For $\mathrm{D} \in \{\mathrm{sSet}_+, \mathrm{sSet}\}$ *and* $w \in \{0,1\}^\omega$ *(*sSet*: only* 0^ω *or* $0^m 1^\omega$ *allowed):*

$$j^w = (j_n^{w_0 \cdots w_n})_{n \in \mathbb{N}} : \Omega \to \Omega \text{ is a topology on } \mathrm{D}.$$

Moreover, every topology on D *arises this way.*

Separated elements in sSet

Notation

For all simplicial sets B, and $\vec{x} = (x_k, \ldots, x_0) \in B(k-1)^{k+1}$,

$$d^{-1}(\vec{x}) \coloneqq \text{all } k\text{-simplices with incidence tuple } \vec{x}.$$

Thus, parallel simplices = distinct elements of $d^{-1}(\vec{x})$.

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Definition

We say that $B \in \{\mathrm{sSet}_+^{\leqslant n}, \mathrm{sSet}^{\leqslant n}, \mathrm{sSet}_+, \mathrm{sSet}\}$ is

- 0-***simple*** if $|B(0)| \leqslant 1$,
- 0-***complete*** if $|B(0)| \geqslant 1$,
- 0-***exact*** if $B(0) = \{\cdot\}$,
- k-***simple*** if $\forall \vec{x}, |d^{-1}(\vec{x})| \leqslant 1$,
- k-***complete*** if $\forall \vec{x}, |d^{-1}(\vec{x})| \geqslant 1$,
- k-***exact*** if $\forall \vec{x}, |d^{-1}(\vec{x})| = 1$.

Example

0-complete $\iff$ nonempty graph
1-simple $\iff$ no parallel edges (simple graph)
2-simple $\iff$ no parallel triangles
...

Separated elements in sSet (continued)

Theorem

Topology j^w on $\mathsf{D} \in \{\mathsf{sSet}_+^{\leqslant n}, \mathsf{sSet}^{\leqslant n}, \mathsf{sSet}_+, \mathsf{sSet}\}$ and object $B \in \mathsf{D}$.

1. *B is j^w-separated $\iff$ B is k-simple for all k with $w_k = 1$.*
2. *B is j^w-complete $\iff$ B is k-complete for all k with $w_k = 1$.*
3. *B is a j^w-sheaf $\iff$ B is k-exact for all k with $w_k = 1$.*

If $D \in \{\mathsf{sSet}_+, \mathsf{sSet}\}$, k ranges over $\mathbb{N}$. Otherwise, k ranges over $\{0, \dots, n\}$.

$\implies$ New quasitoposes identified.

Conclusion

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Definition

$$\left.\begin{array}{r}\text{Sets}\\\text{Graphs}\\\dots\end{array}\right\}\text{Presheaves}$$

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$$\left.\begin{array}{r}\text{Sets}\\ \text{Graphs}\\ \dots\end{array}\right\}\text{Presheaves} \quad\Longrightarrow\quad \left.\begin{array}{r}\text{Fuzzy sets}\\ \text{Fuzzy graphs}\\ \dots\end{array}\right\}\textit{Fuzzy Presheaves}$$

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Theorem

- *Fuzzy presheaves form (rm-adhesive) quasitoposes.*
- *Partially simple graphs form a quasitopos.*
- *For each bit string w, the following two categories are quasitoposes*
  - *the category of simplicial sets k-simple whenever* $w_k = 1$,
  - *the category of simplicial sets k-exact whenever* $w_k = 1$,

Future work

Future work:

- Define “fuzzy categories”? Are they quasitoposes?
- Obtain more quasitoposes via LT-topologies.
- Look at symmetric simplicial sets

Open question: Conegation

Negation $\neg a$ is largest element with $a \wedge \neg a = \bot$.

Open question: Conegation

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A_1: $\boxed{a \quad b}$ $\quad$ $\sim A_1 = A$: $\boxed{a \rightrightarrows b \leftarrow c \circlearrowleft}$ $\quad$ $\sim\sim A_1$: $\boxed{}$

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Conegation seem to be more of an *interior* operation.

- Co-topology?
- Capture local falsehood (dual axioms)?
- Can we obtain other new quasitoposes?