

Composite Theories and Distributive Laws

Aloïs Rosset

joint work with Maaïke Zwart, Helle Hvid Hansen, and Jörg Endrullis

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This talk

Monads \iff Algebraic Theories

Distributive laws \iff Composite Theories

Cheng'20
Piróg, Staton'17

This talk

Monads \longleftrightarrow Algebraic Theories

Distributive laws $\overset{\text{Cheng}'20}{\underset{\text{Piróg, Staton}'17}{\longleftrightarrow}}$ Composite Theories

Weak distributive laws $\overset{?}{\longleftrightarrow}$ *Weak composite theories?*



This talk

Monads \iff Algebraic Theories

Distributive laws \implies Composite Theories
Zwart'20



Weak distributive laws $\overset{?}{\iff}$ *Weak composite theories?*

Preliminaries

Monads

Monads are

- functor $S : C \rightarrow C$
- unit $\eta : id \Rightarrow S$
- multiplication $\mu : SS \Rightarrow S$

$$\begin{array}{ccc}
 S & \xrightarrow{S\eta} & S^2 \\
 \eta^S \downarrow & \parallel & \downarrow \mu \\
 S^2 & \xrightarrow{\mu} & S
 \end{array}
 \qquad
 \begin{array}{ccc}
 S^3 & \xrightarrow{\mu^S} & S^2 \\
 S\mu \downarrow & & \downarrow \mu \\
 S^2 & \xrightarrow{\mu} & S
 \end{array}$$

S-algebras are

- $(X, SX \xrightarrow{\alpha} X)$

$$\begin{array}{ccc}
 X & \xrightarrow{\eta_X} & SX \\
 \parallel & & \downarrow \alpha \\
 & & X
 \end{array}
 \qquad
 \begin{array}{ccc}
 S^2X & \xrightarrow{\mu_X} & SX \\
 S\alpha \downarrow & & \downarrow \alpha \\
 SX & \xrightarrow{\alpha} & X
 \end{array}$$

Distributive laws are

- $\lambda : ST \Rightarrow TS$

$$\begin{array}{ccc}
 & T & \\
 \eta^{ST} \swarrow & & \searrow T\eta^S \\
 ST & \xrightarrow{\lambda} & TS
 \end{array}$$

$$\begin{array}{ccc}
 & S & \\
 S\eta^T \swarrow & & \searrow \eta^{TS} \\
 ST & \xrightarrow{\lambda} & TS
 \end{array}$$

$$\begin{array}{ccc}
 SST & \xrightarrow{S\lambda} & STS & \xrightarrow{\lambda^S} & TSS \\
 \downarrow \mu^{ST} & & & & T\mu^S \downarrow \\
 ST & \xrightarrow{\lambda} & TS
 \end{array}$$

$$\begin{array}{ccc}
 STT & \xrightarrow{\lambda^T} & TST & \xrightarrow{T\lambda} & TTS \\
 \downarrow S\mu^T & & & & \mu^T S \downarrow \\
 ST & \xrightarrow{\lambda} & TS
 \end{array}$$

Algebraic theories

Algebraic theories \mathbb{S} are

- *signature* $\Sigma_{\mathbb{S}} = \{f^{(2)}, g^{(1)}, \dots\}$
- *equations* $E_{\mathbb{S}} = \{(s, t), \dots\}$

\mathbb{S} -*algebras* are $(X, \{X^2 \xrightarrow{f} X, \dots\})$ satisfying equations:

$$[[s]]_{\sigma} = [[t]]_{\sigma} \quad \forall (s, t) \in E, \forall \text{var. assign. } \sigma$$

$$\text{Set} \begin{array}{c} \xrightarrow{\mathcal{T}(\Sigma_{\mathbb{S}}, -)/E_{\mathbb{S}}} \\ \perp \\ \xleftarrow{U} \end{array} \text{Alg}(\mathbb{S}) \implies \text{free algebra monad } T_{\mathbb{S}}$$

Algebraic presentation

\mathbb{S} is an *algebraic presentation* of S if

$$T_{\mathbb{S}} \cong S \quad \text{or equivalently} \quad \mathbf{Alg}(\mathbb{S}) \cong_{\text{conc}} \mathbf{EM}(S)$$

For instance

list	$[x_1, \dots, x_n]$	\iff	$x_1 \cdot \dots \cdot x_n$	in Monoid
set	$\{x_1, \dots, x_n\}$	\iff	$x_1 \cdot \dots \cdot x_n$	in JoinSemiLattices
distribution	$px + (1 - p)y$	\iff	$x \oplus_p y$	in ConvexAlgebras
\vdots		\iff		\vdots

Rewriting

String rewriting

Term rewriting

$$ab \rightarrow c$$

$$f(x, g(y)) \rightarrow f(x, x)$$

$$bc \rightarrow a$$

$$h(x) \rightarrow f(g(x), g(x))$$

$$ca \rightarrow a$$

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Termination (SN)

Local Confluence (WCR)

Confluence (CR)

$$\cdot \rightarrow \cdot \rightarrow \dots \rightarrow \cdot \not\rightarrow$$



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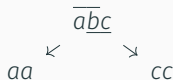
$$\cdot \rightarrow \cdot \rightarrow \dots \rightarrow \cdot \not\rightarrow$$



■ $SN \wedge CR \implies$ terms rewrite to unique normal forms (no more steps).

■ **Newman's Lemma:** $SN \wedge WCR \implies CR$

■ **Critical pairs** are rules that overlap



■ **Critical Pair's Lemma:** $WCR \iff$ all critical pairs converge.

Composite theories

Example

Example: Monoids, Abelian group, and Rings.

$$\Sigma_{\text{Mon}} := \{ \cdot^{(2)}, 1^{(0)} \}$$

$$\Sigma_{\text{AbGrp}} := \{ 0^{(0)}, +^{(2)}, -^{(1)} \}$$

$$E_{\text{Mon}} := \left\{ \begin{array}{l} (x \cdot y) \cdot z = x \cdot (y \cdot z), \\ 1 \cdot x = x, \\ x \cdot 1 = x \end{array} \right\}$$

$$E_{\text{AbGrp}} := \left\{ \begin{array}{l} (x + y) + z = x + (y + z), \\ x + (-x) = 0, \\ x + y = y + x, \\ x + 0 = x \end{array} \right\}$$

Then:

$$\Sigma_{\text{Ring}} := \Sigma_{\text{Mon}} \uplus \Sigma_{\text{AbGrp}} \quad E_{\text{Ring}} := E_{\text{Mon}} \cup E_{\text{AbGrp}} \cup \left\{ \begin{array}{l} x(y + z) = (xy) + (xz), \\ (y + z)x = (yx) + (zx) \end{array} \right\}.$$

We can distribute everything, so every Ring term is equal to an AbGrp term with Monoid terms substituted.

Composite Theories of \mathbb{T} after \mathbb{S}

Definition

Algebraic theories $\mathbb{S}, \mathbb{T} \subseteq \mathbb{U}$.

- \mathbb{U} -term is *separated* if of the form $t[s_x/x]$.
- Two separated terms $t[s_x]$ and $t'[s'_y]$ are *equal modulo* (\mathbb{S}, \mathbb{T}) if

$$\overline{t[s_x]}^{\mathbb{S}, \mathbb{T}} = \overline{t'[s'_y]}^{\mathbb{S}, \mathbb{T}} \quad (\text{in } TSV)$$

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- \mathbb{U} is a *composite theory* of \mathbb{T} after \mathbb{S} if
 - ▶ every \mathbb{U} -term u has a *separation* $u =_{\mathbb{U}} t[s_x/x]$
 - ▶ any $t[s_x] =_{\mathbb{U}} t'[s'_x] \implies t[s_x]$ and $t'[s'_x]$ must be equal modulo (\mathbb{S}, \mathbb{T}) .

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Example of equal modulo (\mathbb{S}, \mathbb{T}) : $0^{\text{AbGrp}} = \overline{(1 \cdot x^{\text{Mon}}) + (- (x \cdot 1^{\text{Mon}}))}^{\text{AbGrp}}$

- $x + (-x) =_{\text{AbGrp}} 0$
- $x \cdot 1 =_{\text{Monoid}} 1 \cdot x$

Dist. Laws \iff Composite Th.

\iff proof**Theorem (D.L \iff Comp. Th. Zwart'20)**

Monads S, T presented by theories \mathbb{S}, \mathbb{T} .

Given composite theory \mathbb{U} of \mathbb{T} after \mathbb{S} , then

$$\lambda_{\mathcal{V}} : ST\mathcal{V} \rightarrow TS\mathcal{V} :$$

$$\overline{s[t_x^T/x]^S} \mapsto \overline{t'[s_x^S/x]^T} \text{ (a separation)}$$

is a distributive law with monad $T \circ_\lambda S$ presented by \mathbb{U} .

Proof.

λ well-defined by equality modulo (\mathbb{S}, \mathbb{T}) .

Straightforward but tedious. □

\implies proof**Theorem (D.L \implies Comp. Th.)***Monads S, T presented by theories \mathbb{S}, \mathbb{T} .**Distributive law $\lambda : ST \Rightarrow TS$.*

$$E_\lambda := \left\{ (s[t_x/x], t[s_y/y]) \mid \lambda_v(\overline{s[t_x^T/x]^S}) = \overline{t[s_y^S/y]^T} \right\}.$$

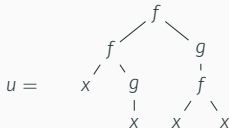
$$\Sigma_{\mathbb{U}^\lambda} := \Sigma_{\mathbb{S}} \uplus \Sigma_{\mathbb{T}},$$

$$E_{\mathbb{U}^\lambda} := E_{\mathbb{S}} \cup E_{\mathbb{T}} \cup E_\lambda.$$

*Then, \mathbb{U}^λ is a composite theory of \mathbb{T} after \mathbb{S} .*More **tools** needed for the proof.

Tools needed

- Define function type that give U^λ -terms a corresponding $\{S, T\}^* \mathcal{V}$.
E.g. for $f^{(2)} \in \Sigma_S$, and $g^{(1)} \in \Sigma_T$:

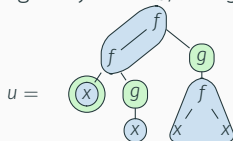


$\text{type}(u) := STS\mathcal{V}$.

$$\bar{u}^{STS} := \overline{\overline{\overline{f(x^S, g(x^S))}}_{S^T} \overline{g(f(x, x))}_{S^T}}_{S^T}.$$

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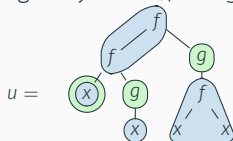


$\text{type}(u) := STSV.$

$$\bar{u}^{STS} := \overline{f(f(\bar{x}^{ST}, g(\bar{x}^{ST})), g(f(x, x))^{ST})}.$$

Tools needed

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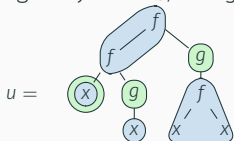
$\text{type}(u) := STSV.$

$$\bar{u}^{STS} := \overline{f(f(\bar{x}^{ST}, g(\bar{x}^{ST})), g(f(x, x)^{ST}))}.$$

- Apply λ, μ^S, μ^T to \bar{u}^{STS} until we reach TSV, TV or SV

Tools needed

- Define function type that give \mathbb{U}^λ -terms a corresponding $\{S, T\}^* \mathcal{V}$.
E.g. for $f^{(2)} \in \Sigma_S$, and $g^{(1)} \in \Sigma_T$:



$\text{type}(u) := STSV.$

$$\bar{u}^{STS} := \frac{}{f(f(\bar{x}^{\bar{S}^T}, g(\bar{x}^{\bar{S}^T})), g(f(x, x))^{\bar{S}^T})^{\bar{S}}}$$

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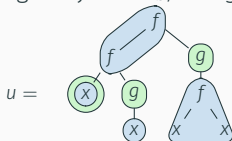
Definition (c.f. *rewrite category* Kozen'19)

Functor rewriting system (FRS) (Σ, \mathcal{R}) consist of

- $\Sigma := \{F_i \mid i \in I\}$, set of functors
- $\mathcal{R} := \{\alpha_j : w_j \rightarrow w'_j \mid w_j, w'_j \in \Sigma^*, j \in J\}$, set of natural transformations

Tools needed

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E.g. for $f^{(2)} \in \Sigma_S$, and $g^{(1)} \in \Sigma_T$:



$\text{type}(u) := STSV.$

$$\bar{u}^{STS} := \frac{}{\overline{\overline{\overline{f(f(\bar{x}^{ST}, g(\bar{x}^{ST}))}, g(\overline{\overline{\overline{f(x, x)}^{ST}})^{ST}})^{ST}})^{ST}}}$$

- Apply λ, μ^S, μ^T to \bar{u}^{STS} until we reach TSV, TV or SV

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We define: $\mathcal{R}^{sep} = (\Sigma, R)$, where

- $\Sigma := \{S, T\}$
- $R := \{\lambda : ST \rightarrow TS, \mu^S : SS \rightarrow S, \mu^T : TT \rightarrow T\}$

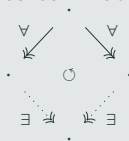
Properties of FRS

Definition

Local Confluence-commuting (WCR ○)



Confluence-commuting (CR ○)



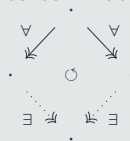
Properties of FRS

Definition

Local Confluence-commuting (WCR ○)



Confluence-commuting (CR ○)



Lemma (FRS Newman's Lemma)

$SN \wedge WCR \iff CR$

Lemma (FRS Critical Pair's Lemma)

$WCR \iff$ all critical pairs converge with a commuting diagram.

Properties of \mathcal{R}^{sep}

Lemma

$\mathcal{R}^{sep} = (\{S, T\}, \{\lambda, \mu^S, \mu^T\})$ is SN and CR ○.

Proof.

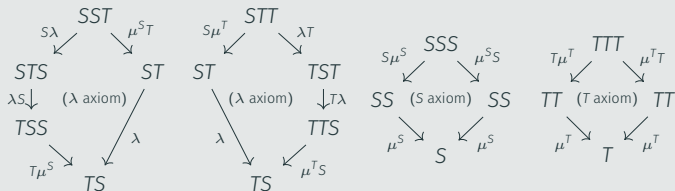
- SN: polynomial interpretation over \mathbb{N} . $\llbracket S \rrbracket(x) := 2x + 1$, $\llbracket T \rrbracket(x) := x + 1$

$$\llbracket ST \rrbracket(x) = 2x + 3 > 2x + 2 = \llbracket TS \rrbracket(x),$$

$$\llbracket SS \rrbracket(x) = 4x + 3 > 2x + 1 = \llbracket S \rrbracket(x),$$

$$\llbracket TT \rrbracket(x) = x + 2 > x + 1 = \llbracket T \rrbracket(x).$$

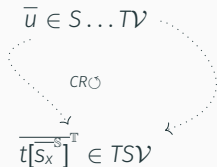
- WCR○: exactly 4 critical pairs:



Consequences of \mathcal{R}^{sep} being SN and CR

For \mathbb{U}^λ -term u , define $sep(u)$ separated and $u =_{\mathbb{U}^\lambda} sep(u)$, thanks to:

- Unique normal form (TS , S or T)
- Any paths to normal form are equal.



Lemma

Every \mathbb{U}^λ -term can be separated. ✓

Finishing the proof

Lemma

Any two separated terms equal in \mathbb{U}^λ are equal modulo (\mathbb{S}, \mathbb{T}) .

Sketch of proof.

Induction on proof-tree.

Each $u = u'$, we prove $\text{sep}(u), \text{sep}(u')$ are equal modulo (\mathbb{S}, \mathbb{T})

$\frac{(s, t) \in E_{\mathbb{S}}}{s = t} \text{ Ax.}$	E.g. $\overline{\text{sep}(s_1)}^{\mathbb{S}} = \overline{s_1}^{\mathbb{S}} = \overline{s_2}^{\mathbb{S}} = \overline{\text{sep}(s_2)}^{\mathbb{S}}$
$\frac{}{u = u} \text{ Refl.}$	$\overline{\text{sep}(u)}^{\mathbb{T}\mathbb{S}} = \overline{\text{sep}(u)}^{\mathbb{T}\mathbb{S}}$
$\frac{u_1 = u_2}{u_2 = u_1} \text{ Sym.}$	IH = goal

Continued

Sketch of proof continued.

$$\frac{u_1 = u_2 \quad u_2 = u_3}{u_1 = u_3} \text{Trans.}$$

$$\overline{\text{sep}(u_1)}^{TS} = \overline{\text{sep}(u_2)}^{TS} = \overline{\text{sep}(u_3)}^{TS}$$

$$\frac{u_1 = u'_1 \quad \dots \quad u_n = u'_n}{\text{op}(u_1, \dots, u_n) = \text{op}(u'_1, \dots, u'_n)} \text{Cong.}$$

E.g. when $\text{op} \in \Sigma_T$:

$$\begin{aligned} & \overline{\text{sep}(\text{op}(u_1, \dots, u_n))}^{TS} \\ &= \mu^T S(\overline{\text{op}(t_1[s_1], \dots, t_n[s_n])}^{TTS}) \\ &\stackrel{\text{IH}}{=} \mu^T S(\overline{\text{op}(t'_1[s'_1], \dots, t'_n[s'_n])}^{TTS}) \\ &= \overline{\text{sep}(\text{op}(u'_1, \dots, u'_n))}^{TS} \end{aligned}$$

$$\frac{u = u'}{u[f] = u'[f]} \text{Subst.}$$

Separate substitution: $f(y) =_{U^\lambda} t_y[s_z]$.

$$\begin{aligned} & \overline{\text{sep}(u[f])}^{TS} \\ &= \mu^T S(\overline{(t_{[s_x][t_y[s_z]]}]^{TSTS})} \\ &\stackrel{\text{IH}}{=} \mu^T S(\overline{(t'_{[s'_x][t_y[s_z]]}]^{TSTS})} \\ &= \overline{\text{sep}(u'[f])}^{TS} \end{aligned}$$



Presentation of \mathbb{U}^λ

Theorem (Zwart'20)

The monad $T \circ_\lambda S$ is presented by \mathbb{U}^λ .

Proof updated.

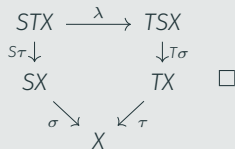
Shortcut $\mathbf{EM}(TS) \cong_{\text{conc}} \mathbf{Alg}(\lambda)$.

λ -algebras are triples (X, σ, τ) , such that

- (X, σ) is an S -algebra
- (X, τ) is a T -algebra

λ -algebra morphisms are $f: X \rightarrow Y$ such that

- $f: (X, \sigma_X) \rightarrow (Y, \sigma_Y)$ is S -algebra morphism.
- $f: (X, \tau_X) \rightarrow (Y, \tau_Y)$ is T -algebra morphisms.



Axiomatisation \mathbb{U}^λ

Axiomatisation example

Main theorem requires E_λ to contain **all** distributivity equations.

Example (Ring)

$$\lambda : \text{Mon} \cdot \text{AbGrp} \Rightarrow \text{AbGrp} \cdot \text{Mon}$$

$x = x$	$(x + y)z = xz + yz$	$x \cdot 0 = 0$	$(-x)y = -(xy)$
$x = x + 0$	$x(y + z) = xy + xz$	$0 \cdot x = 0$	$x(-y) = -(xy)$
$x = 0 + x$	$(x + y)(z + w) = xz + xw + yz + yw$	$0 \cdot x = 0 + 0$	$(-x)(-y) = xy$
\vdots	\vdots	\vdots	\vdots

Goal: Find **minimal** axiomatisation \implies general tools

Layers

Definition

ST-layers of term $s[t_x/x] \in \Sigma_{\mathbb{S}}^* \Sigma_{\mathbb{T}}^* \mathcal{V}$, are pair $(m, n) \in \mathbb{N}^2$

$$\begin{cases} m := \text{depth}(s) \\ n := \max\{\text{depth}(t_x) \mid x \in \text{var}(s)\} \end{cases} \quad (\text{const. depth } 1)$$

Example (Ring, $\mathbb{S} = \text{Mon}$, $\mathbb{T} = \text{AbGrp}$)

ST-Layers	(0, 0)	(0, 1)	(1, 0)	(1, 1)	(0, 2)
Examples	x	0	1	$x \cdot 0$	$x + 0$
	y	$x + y$	$x \cdot y$	$(x + y) \cdot (y + z)$	$(x + y) + z$

Lemmas

Lemma

For all $E' \subseteq E_\lambda$ such that for each $f^{(n)} \in \Sigma_S$, $g^{(m)} \in \Sigma_T$ and each $i \in \{1, \dots, n\}$, E' contains one equation of the form $l = r$, where

- $l = f(x_1, \dots, x_{i-1}, g(\vec{y}), x_{i+1}, \dots, x_n)$
- $r \in \lambda_V(\vec{l}^{ST})$,

If the TRS $(\Sigma_{U^\lambda} = \Sigma_S \uplus \Sigma_T, E')$ is terminating (SN), then

congruence by $E_S \cup E_T \cup E' =$ congruence by $E_S \cup E_T \cup E_\lambda$.

Lemma

If R is a set rules of the form $s[t_x/x] \rightarrow t[s_y/y]$ such that

- $s[t_x/x]$ has ST-layers $(1, 1)$
- $t[s_y/y]$ has TS-layers $(*, 1)$
- s_y is linear¹ in $Z = \{t_x \mid t_x \text{ is a variable}\}$,

then R is terminating.

¹Linear in a TRS sense, i.e. variables appearing at most once.

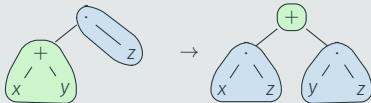
Axiomatisation examples

Example

- Ring from $\lambda : \text{Mon} \cdot \text{AbGrp} \rightarrow \text{AbGrp} \cdot \text{Mon}$.

$$(x + y)z = xz + yz:$$

- ▶ RHS TS-layers (1, 1) ✓
- ▶ linearity ✓



- $\lambda : \mathcal{DR} \rightarrow \mathcal{RD}$ (distribution over reader): for each $p \in [0, 1]$

$$f(x_1, \dots, x_n) \oplus_p y = f(x_1 \oplus_p y, \dots, x_n \oplus_p y).$$

- $\lambda : \mathcal{MD} \rightarrow \mathcal{DM}$ (multiset over distribution): for each $p \in [0, 1]$

$$(x_1 \oplus_p x_2) \cdot y = (x_1 \cdot y) \oplus_p (x_2 \cdot y).$$

- $\lambda : \text{Mon}^+ \text{Mon}^+ \rightarrow \text{Mon}^+ \text{Mon}^+$ (non-empty list over itself)

$$a * (b \star c) = a * b$$

$$(a \star b) * c = a * c.$$

Counterexample

Note: $E' \subseteq E_\lambda$ not terminating \implies conclusion not guaranteed.

Example (Famous $ab \rightarrow bbaa$ TRS example)

Define two theories and a distributive law:

$$\left\{ \begin{array}{l} \Sigma_S := \{a^{(1)}\} \\ E_S := \{aaa = aa\} \end{array} \right. \quad \lambda: ST\mathcal{V} \rightarrow T\mathcal{S}\mathcal{V} \quad \begin{array}{l} \xrightarrow{\overline{\overline{a^n b^m x}}^{TS}} \\ \xrightarrow{\overline{\overline{a^n x}}^{TS}} \end{array} \quad \begin{array}{l} \rightarrow T\mathcal{S}\mathcal{V} \\ \xrightarrow{\overline{\overline{b^2 a^2 x}}^{ST}} \\ \xrightarrow{\overline{\overline{a^n x}}^{ST}} \end{array} \quad \begin{array}{l} , \text{ for } n, m \in \{1, 2\} \\ , \text{ for } n \in \{1, 2\} \end{array}$$

$$\left\{ \begin{array}{l} \Sigma_T := \{b^{(1)}\} \\ E_T := \{bbb = bb\} \end{array} \right. \quad \begin{array}{l} \xrightarrow{\overline{\overline{b^n x}}^{TS}} \\ \xrightarrow{\overline{\overline{x}}^{TS}} \end{array} \quad \begin{array}{l} \xrightarrow{\overline{\overline{b^n x}}^{ST}} \\ \xrightarrow{\overline{\overline{x}}^{ST}} \end{array} \quad \begin{array}{l} , \text{ for } n \in \{1, 2\} \\ \end{array}$$

However $E' = \{ab = b^2 a^2\}$ cannot derive $(aab, bbaa) \in E_\lambda$.

$$\begin{aligned} \underline{aab} &=_{E'} \underline{abb} \underline{aa} =_{E'} \underline{bba} \underline{ab} \underline{aa} =_{E'} \underline{bbab} \underline{baaa} =_{E_S} \underline{bbab} \underline{baa} \\ &=_{E'} \underline{bbb} \underline{baab} \underline{aa} =_{E_T} \underline{bba} \underline{ab} \underline{aa} =_{E'} \dots \text{ (loop)} \end{aligned}$$

Conclusion

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Contribution:

- Proved constructively: Distributive Laws \iff Composite Theories.
More than the result: it's the proof strategy.
- Gave criteria for minimal axiomatisation $E' \subseteq E_\lambda$.

Future work:

- More TRS criteria for $E' \subseteq E_\lambda$ termination.
- Extend correspondence further:
 - ▶ *weak composite theories?*
 - ▶ *multi-sorted distributive laws?*