Composite Theories and Distributive Laws

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VRIJE UNIVERSITEIT AMSTERDAM

This talk

Monads $\Longleftrightarrow$ Algebraic Theories

Distributive laws $\overset{\text{Cheng'20}}{\underset{\text{Piróg, Staton'17}}{\Longleftrightarrow}}$ Composite Theories

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Weak distributive laws $\overset{?}{\Longleftrightarrow}$ *Weak composite theories?*

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Distributive laws $\underset{\text{Zwart'20}}{\overset{\Longrightarrow}{\Longleftarrow}}$ Composite Theories

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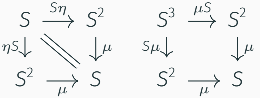
Preliminaries

Monads

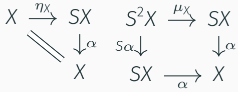
Monads are

- functor $S : C \to C$
- unit $\eta : id \Rightarrow S$
- multiplication $\mu : SS \Rightarrow S$



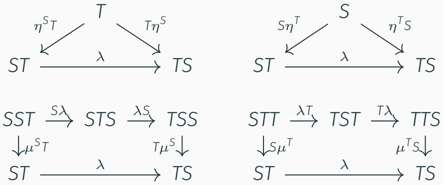
S-algebras are

- $(X, SX \xrightarrow{\alpha} X)$



Distributive laws are

- $\lambda : ST \Rightarrow TS$



Algebraic theories

Algebraic theories $\mathbb{S}$ are

- ***signature*** $\Sigma_{\mathbb{S}} = \{f^{(2)}, g^{(1)}, \ldots\}$
- ***equations*** $E_{\mathbb{S}} = \{(s,t), \ldots\}$

$\mathbb{S}$-***algebras*** are $(X, \{X^2 \xrightarrow{[\![f]\!]} X, \ldots\})$ with

$$[\![s]\!]_\sigma = [\![t]\!]_\sigma \qquad \forall (s,t) \in E, \forall \text{var. assign. } \sigma$$

$$\mathrm{Set} \underset{U}{\overset{\mathrm{Alg}(\mathbb{S})}{\rightleftarrows}} \mathrm{Alg}(\mathbb{S}) \implies \textit{free algebra monad } T_{\mathbb{S}}$$

Algebraic presentation

$\mathbb{S}$ is an ***algebraic presentation*** of S if

$$T_{\mathbb{S}} \cong S \qquad \text{or equivalently} \qquad \mathbf{Alg}(\mathbb{S}) \cong_{\text{conc}} \mathbf{EM}(S)$$

For instance

$$\begin{array}{rccll} \text{list} & [x_1, \ldots, x_n] & \Longleftrightarrow & x_1 \cdot \ldots \cdot x_n & \text{in Monoid} \\ \text{set} & \{x_1, \ldots, x_n\} & \Longleftrightarrow & x_1 \cdot \ldots \cdot x_n & \text{in JoinSemiLattices} \\ \text{distribution} & px + (1-p)y & \Longleftrightarrow & x \oplus_p y & \text{in ConvexAlgebras} \\ \vdots & & \Longleftrightarrow & & \vdots \end{array}$$

Rewriting

String rewriting	Term rewriting
$ab \to c$	$f(x, g(y)) \to f(x, x)$
$bc \to a$	$h(x) \to f(g(x), g(x))$
$ca \to a$	

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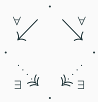
Termination (SN)	Local Confluence (WCR)	Confluence (CR)
$\cdot \to \cdot \to \ldots \to \cdot \not\to$		

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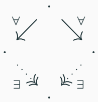
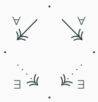
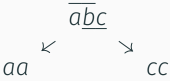
$\cdot \to \cdot \to \ldots \to \cdot \not\to$		

- $\mathsf{SN} \wedge \mathsf{CR} \implies$ terms rewrite to unique normal forms (no more steps).
- **Newman's Lemma:** $SN \wedge WCR \implies CR$
- *Critical pairs* are rules that overlap 

- **Critical Pair's Lemma:** $WCR \iff$ all critical pairs converge.

Composite theories

Example

Example: Monoids, Abelian group, and Rings.

$$\Sigma_{\text{Mon}} := \{\cdot^{(2)}, 1^{(0)}\} \qquad \Sigma_{\text{AbGrp}} := \{0^{(0)}, +^{(2)}, -^{(1)}\}$$

$$E_{\text{Mon}} := \left\{ \begin{array}{c} (x \cdot y) \cdot z = x \cdot (y \cdot z), \\ 1 \cdot x = x, \\ x \cdot 1 = x \end{array} \right\} \qquad E_{\text{AbGrp}} := \left\{ \begin{array}{c} (x + y) + z = x + (y + z), \\ x + (-x) = 0, \\ x + y = y + x, \\ x + 0 = x \end{array} \right\}$$

Then:

$$\Sigma_{\text{Ring}} := \Sigma_{\text{Mon}} \uplus \Sigma_{\text{AbGrp}} \qquad E_{\text{Ring}} := E_{\text{Mon}} \cup E_{\text{AbGrp}} \cup \left\{ \begin{array}{c} x(y+z) = (xy) + (xz), \\ (y+z)x = (yx) + (zx) \end{array} \right\}.$$

We can distribute everything, so every Ring term is equal to an AbGrp term with Monoid terms substituted.

Composite Theories of $\mathbb{T}$ after $\mathbb{S}$

Definition

Algebraic theories $\mathbb{S}, \mathbb{T} \subseteq \mathbb{U}$.

- $\mathbb{U}$-term is ***separated*** if of the form $t[s_x/x]$.
- Two separated terms $t[s_x]$ and $t'[s'_y]$ are ***equal modulo*** $(\mathbb{S}, \mathbb{T})$ if

$$\overline{t[\overline{s_x}^{\mathbb{S}}]}^{\mathbb{T}} = \overline{t'[\overline{s'_y}^{\mathbb{S}}]}^{\mathbb{T}} \quad \text{(in } TS\mathcal{V}\text{)}$$

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  - every $\mathbb{U}$-term u has a ***separation*** $u =_{\mathbb{U}} t[s_x/x]$
  - any $t[s_x] =_{\mathbb{U}} t'[s'_x] \implies t[s_x]$ and $t'[s'_x]$ must be equal modulo $(\mathbb{S}, \mathbb{T})$.

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Example of equal modulo $(\mathbb{S}, \mathbb{T})$: $\overline{0}^{\text{AbGrp}} = \overline{(\overline{1 \cdot x}^{\text{Mon}}) + (-(\overline{x \cdot 1}^{\text{Mon}}))}^{\text{AbGrp}}$.

- $x + (-x) =_{\text{AbGrp}} 0$
- $x \cdot 1 =_{\text{Monoid}} 1 \cdot x$

Dist. Laws $\Longleftrightarrow$ Composite Th.

$\Longleftarrow$ proof

Theorem (D.L $\Longleftarrow$ Comp. Th. Zwart'20)

Monads S, T presented by theories $\mathbb{S}, \mathbb{T}$.
Given composite theory $\mathbb{U}$ of $\mathbb{T}$ after $\mathbb{S}$, then

$$\lambda_{\mathcal{V}} : ST\mathcal{V} \to TS\mathcal{V} :$$
$$\overline{s[\overline{t_x}^{\mathbb{T}}/x]}^{\mathbb{S}} \mapsto \overline{t'[\overline{s'_x}^{\mathbb{S}}/x]}^{\mathbb{T}} \text{ (a separation)}$$

is a distributive law with monad $T \circ_\lambda S$ presented by $\mathbb{U}$.

Proof.

λ well-defined by equality modulo $(\mathbb{S}, \mathbb{T})$.
Straightforward but tedious. □

$\Longrightarrow$ proof

Theorem (D.L $\Longrightarrow$ Comp. Th.)

Monads S, T presented by theories $\mathbb{S}, \mathbb{T}$.
Distributive law $\lambda : ST \Rightarrow TS$.

$$E_\lambda := \left\{ \left(s[t_x/x], t[s_y/y]\right) \mid \lambda_\mathcal{V}\left(\overline{s[\overline{t_x}^{\mathbb{T}}/x]}^{\mathbb{S}}\right) = \overline{t[\overline{s_y}^{\mathbb{S}}/y]}^{\mathbb{T}} \right\}.$$

$$\Sigma_{\mathbb{U}^\lambda} := \Sigma_\mathbb{S} \uplus \Sigma_\mathbb{T},$$

$$E_{\mathbb{U}^\lambda} := E_\mathbb{S} \cup E_\mathbb{T} \cup E_\lambda.$$

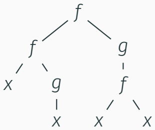
Then, $\mathbb{U}^\lambda$ is a composite theory of $\mathbb{T}$ after $\mathbb{S}$.

More tools needed for the proof.

Tools needed

Idea:

- See $\mathbb{U}^\lambda$-terms in $\{S,T\}^*\mathcal{V}$, e.g. for $f^{(2)} \in \Sigma_{\mathbb{S}}$, and $g^{(1)} \in \Sigma_{\mathbb{T}}$:



$$\text{type}(u) := STS\mathcal{V}.$$

$$\overline{u}^{STS} := \overline{f(f(\overline{\overline{x}^{\mathbb{S}}}^{\mathbb{T}}, \overline{g(\overline{x}^{\mathbb{S}})}^{\mathbb{T}}), \overline{g(\overline{f(x,x)}^{\mathbb{S}})}^{\mathbb{T}})}^{\mathbb{S}}.$$

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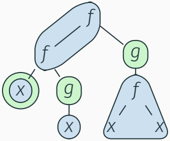

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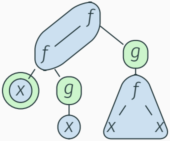

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- Apply λ, μ^S, μ^T to $\overline{u}^{STS}$ until we reach $TS\mathcal{V}$, $T\mathcal{V}$ or $S\mathcal{V}$

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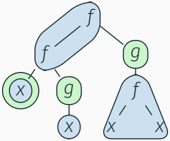

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Definition (c.f. *rewrite category* Kozen'19)

Functor rewriting system (FRS) $(\Sigma, \mathcal{R})$ consist of

- $\Sigma := \{F_i \mid i \in I\}$, set of functors
- $\mathcal{R} := \{\alpha_j : w_j \to w'_j \mid w_j, w'_j \in \Sigma^*, j \in J\}$, set of natural transformations

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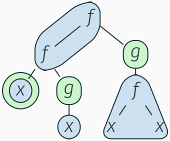

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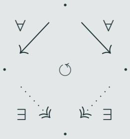
We define: $\mathcal{R}^{sep} = (\Sigma, R)$, where

- $\Sigma := \{S, T\}$
- $R := \{\lambda : ST \to TS,\ \mu^S : SS \to S,\ \mu^T : TT \to T\}$

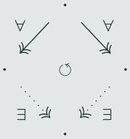
Properties of FRS

Definition

Local Confluence-commuting (WCR⟲)



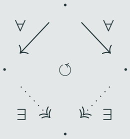
Confluence-commuting (CR ⟲)



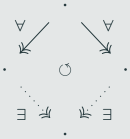
Properties of FRS

Definition

Local Confluence-commuting (WCR↺)



Confluence-commuting (CR ↺)



Lemma (FRS Newman's Lemma)

SN $\wedge$ *WCR* ↺ $\Longrightarrow$ *CR* ↺

Lemma (FRS Critical Pair's Lemma)

WCR ↺ $\Longleftrightarrow$ *all critical pairs converge with commuting diagram.*

Properties of $\mathcal{R}^{sep}$

Lemma

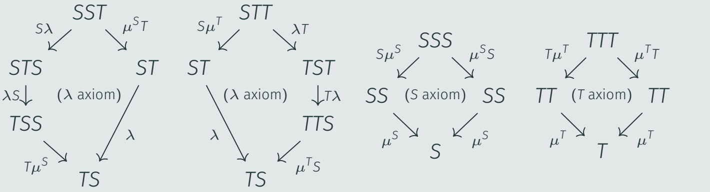
$\mathcal{R}^{sep} = (\{S, T\}, \{\lambda, \mu^S, \mu^T\})$ is SN *and* CR $\circlearrowleft$.

Proof.

- SN: polynomial interpretation over $\mathbb{N}$. $[\![S]\!](x) := 2x + 1$, $[\![T]\!](x) := x + 1$

$$[\![ST]\!](x) = 2x + 3 > 2x + 2 = [\![TS]\!](x),$$
$$[\![SS]\!](x) = 4x + 3 > 2x + 1 = [\![S]\!](x),$$
$$[\![TT]\!](x) = x + 2 > x + 1 = [\![T]\!](x).$$

- WCR$\circlearrowleft$: exactly 4 critical pairs:

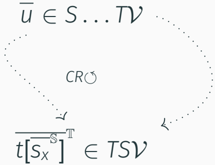


$\square$

Consequences of $\mathcal{R}^{sep}$ being SN and CR ↻

For $\mathbb{U}^\lambda$-term u, define $\mathrm{sep}(u)$ separated and $u =_{\mathbb{U}^\lambda} \mathrm{sep}(u)$, thanks to:

- Unique normal form (TS, S or T)
- Any paths to normal form are equal.



Lemma

Every $\mathbb{U}^\lambda$-term can be separated. ✓

Finishing the proof

Lemma

Any two separated terms equal in $\mathbb{U}^\lambda$ are equal modulo $(\mathbb{S}, \mathbb{T})$.

Sketch of proof.

Induction on proof-tree.

Each $u = u'$, we prove $\mathrm{sep}(u), \mathrm{sep}(u')$ are equal modulo $(\mathbb{S}, \mathbb{T})$

$\dfrac{(s,t) \in E_{\mathbb{S}}}{s = t}$ Ax.	E.g. $\overline{\mathrm{sep}(s_1)}^{\mathbb{S}} = \overline{s_1}^{\mathbb{S}} = \overline{s_2}^{\mathbb{S}} = \overline{\mathrm{sep}(s_2)}^{\mathbb{S}}$
$\dfrac{}{u = u}$ Refl.	$\overline{\mathrm{sep}(u)}^{TS} = \overline{\mathrm{sep}(u)}^{TS}$
$\dfrac{u_1 = u_2}{u_2 = u_1}$ Sym.	IH = goal

Continued

Sketch of proof continued.

$$\frac{u_1 = u_2 \qquad u_2 = u_3}{u_1 = u_3}\ \text{Trans.}$$	$$\overline{\text{sep}(u_1)}^{TS} = \overline{\text{sep}(u_2)}^{TS} = \overline{\text{sep}(u_3)}^{TS}$$
$$\frac{u_1 = u_1' \qquad \dots \qquad u_n = u_n'}{\text{op}(u_1,\dots,u_n) = \text{op}(u_1',\dots,u_n')}\ \text{Cong.}$$	E.g. when $\text{op} \in \Sigma_{\mathbb{T}}$: $$\overline{\text{sep}(\text{op}(u_1,\dots,u_n))}^{TS} = \mu^T S\left(\overline{\text{op}(t_1[s_1],\dots,t_n[s_n])}^{TTS}\right) \overset{\text{IH}}{=} \mu^T S\left(\overline{\text{op}(t_1'[s_1'],\dots,t_n'[s_n'])}^{TTS}\right) = \overline{\text{sep}(\text{op}(u_1',\dots,u_n'))}^{TS}$$
$$\frac{u = u'}{u[f] = u'[f]}\ \text{Subt.}\ .$$	Separate substitution: $f(y) =_{\mathbb{U}^\lambda} t_y[s_z]$. $$\overline{\text{sep}(u[f])}^{TS} = \mu^{TS}\left(\overline{t[s_x[t_y[s_z]]]}^{TSTS}\right) \overset{\text{IH}}{=} \mu^{TS}\left(\overline{t'[s_x'[t_y[s_z]]]}^{TSTS}\right) = \overline{\text{sep}(u'[f])}^{TS}$$

□

Presentation of $\mathbb{U}^\lambda$

Theorem (Zwart'20)

The monad $T \circ_\lambda S$ is presented by $\mathbb{U}^\lambda$.

Proof updated.

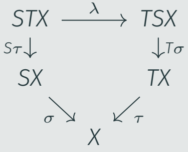
Shortcut $\mathbf{EM}(TS) \cong_{\mathrm{conc}} \mathbf{Alg}(\lambda)$.

λ-algebras are triples (X, σ, τ), such that

- (X, σ) is an S-algebra
- (X, τ) is a T-algebra

λ-algebra morphisms are $f : X \to Y$ such that

- $f : (X, \sigma_X) \to (Y, \sigma_Y)$ is S-algebra morphism.
- $f : (X, \tau_X) \to (Y, \tau_Y)$ is T-algebra morphisms.



□

Axiomatisation $\mathbb{U}^\lambda$

Axiomatisation example

Main theorem requires E_λ to contain all distributivity equations.

Example (Ring)

$$\lambda : \mathrm{Mon} \cdot \mathrm{AbGrp} \Rightarrow \mathrm{AbGrp} \cdot \mathrm{Mon}$$

$$\begin{array}{llll} x = x & (x+y)z = xz + yz & x \cdot 0 = 0 & (-x)y = -(xy) \\ x = x + 0 & x(y+z) = xy + xz & 0 \cdot x = 0 & x(-y) = -(xy) \\ x = 0 + x & (x+y)(z+w) = xz + xw + yz + yw & 0 \cdot x = 0 + 0 & (-x)(-y) = xy \\ \vdots & \vdots & \vdots & \vdots \end{array}$$

Goal: Find minimal axiomatisation $\Longrightarrow$ general tools

Layers

Definition

ST-***layers*** of term $s[t_x/x] \in \Sigma_{\mathbb{S}}^* \Sigma_{\mathbb{T}}^* \mathcal{V}$, are pair (m, n)

$$\begin{cases} m := \mathrm{depth}(s) \\ n := \mathbf{max}\{\mathrm{depth}(t_x) \mid x \in \mathrm{var}(s)\} \end{cases} \qquad \text{(const. depth 1)}$$

Example (Ring, $\mathbb{S} = \mathrm{Mon}$, $\mathbb{T} = \mathrm{AbGrp}$)

ST-Layers	$(0,0)$	$(0,1)$	$(1,0)$	$(1,1)$	$(0,2)$
Examples	x	0	1	$x \cdot 0$	$x + 0$
	y	$x + y$	$x \cdot y$	$(x+y)\cdot(y+z)$	$(x+y)+z$

Lemmas

Lemma

For all $E' \subseteq E_\lambda$ such that for each $f^{(n)} \in \mathbb{S}$, $g^{(m)} \in \mathbb{T}$ and each $i \in \{1, \ldots, n\}$, E' contains one equation of the form $l = r$, where

- $l = f(x_1, \ldots, x_{i-1}, g(\vec{y}), x_{i+1}, \ldots, x_n)$
- $r \in \lambda_{\mathcal{V}}(\overline{l}^{ST})$,

If the TRS $(\Sigma_{\mathbb{U}^\lambda} = \Sigma_{\mathbb{S}} \uplus \Sigma_{\mathbb{T}}, E')$ is terminating, then

$$\text{congruence by } E_{\mathbb{S}} \cup E_{\mathbb{T}} \cup E' = \text{congruence by } E_{\mathbb{S}} \cup E_{\mathbb{T}} \cup E_\lambda.$$

Lemma

If R is a set rules of the form $s[t_x/x] \to t[s_y/y]$ such that

- $s[t_x/x]$ *has ST-layers* $(1, 1)$
- $t[s_y/y]$ *has TS-layers* $(*, 1)$
- s_y *is linear*[1] *in* $Z = \{t_x \mid t_x \text{ is a variable}\}$,

then R is terminating.

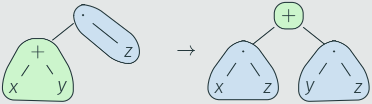
[1] *Linear* in a TRS sense, i.e. variables appearing *at most* once.

Axiomatisation examples

Example

- Ring from λ : Mon · AbGrp $\to$ AbGrp · Mon.

  $(x+y)z = xz + yz$:
  - RHS *TS*-layers (1, 1) ✓
  - linearity ✓



- $\lambda : \mathcal{D}R \to R\mathcal{D}$ (distribution over reader): for each $p \in [0,1]$

$$f(x_1, \ldots, x_n) \oplus_p y = f(x_1 \oplus_p y, \ldots, x_n \oplus_p y).$$

- $\lambda\colon \mathcal{M}\mathcal{D} \to \mathcal{D}\mathcal{M}$ (multiset over distribution): for each $p \in [0,1]$

$$(x_1 \oplus_p x_2) \cdot y = (x_1 \cdot y) \oplus_p (x_2 \cdot y).$$

- $\lambda\colon \mathrm{Mon}^+\mathrm{Mon}^+ \to \mathrm{Mon}^+\mathrm{Mon}^+$ (non-empty list over itself)

$$a * (b \star c) = a * b$$
$$(a \star b) * c = a * c.$$

Counterexample

Note: $E' \subseteq E_\lambda$ not terminating $\implies$ conclusion not guaranteed.

Example (Famous $ab \to bbaa$ TRS example)

Define two theories and a distributive law:

$$\begin{cases} \Sigma_{\mathbb{S}} := \{a^{(1)}\} \\ E_{\mathbb{S}} := \{aaa = aa\} \end{cases}$$

$$\begin{cases} \Sigma_{\mathbb{T}} := \{b^{(1)}\} \\ E_{\mathbb{T}} := \{bbb = bb\} \end{cases}$$

$$\begin{array}{rcll} \lambda \colon ST\mathcal{V} & \to & TS\mathcal{V} & \\ \overline{a^n \overline{b^m \overline{x}}^{\mathbb{T}}}^{\mathbb{S}} & \mapsto & \overline{b^2 \overline{a^2 \overline{x}}^{\mathbb{S}}}^{\mathbb{T}}, & \text{for } n, m \in \{1,2\} \\ \overline{a^n \overline{\overline{x}}^{\mathbb{T}}}^{\mathbb{S}} & \mapsto & \overline{\overline{a^n \overline{x}}^{\mathbb{S}}}^{\mathbb{T}}, & \text{for } n \in \{1,2\} \\ \overline{\overline{b^n \overline{x}}^{\mathbb{T}}}^{\mathbb{S}} & \mapsto & \overline{b^n \overline{\overline{x}}^{\mathbb{S}}}^{\mathbb{T}}, & \text{for } n \in \{1,2\} \\ \overline{\overline{\overline{x}}^{\mathbb{T}}}^{\mathbb{S}} & \mapsto & \overline{\overline{\overline{x}}^{\mathbb{S}}}^{\mathbb{T}} & \end{array}$$

However $E' = \{ab = b^2a^2\}$ cannot derive $(aab, bbaa) \in E_\lambda$.

$$a\underline{ab} =_{E'} \underline{ab}baa =_{E'} bba\underline{ab}aa =_{E'} bbabb\underline{aaaa} =_{E_{\mathbb{S}}} bb\underline{ab}baa$$
$$=_{E'} \underline{bbbb}aabaa =_{E_{\mathbb{T}}} bbaabaa =_{E'} \ldots \text{ (loop)}$$

Conclusion

Conclusion

Contribution:

- Proved constructively: Distributive Laws $\iff$ Composite Theories.
  More than the result: it's the proof strategy.
- Gave criteria for minimal axiomatisation $E' \subseteq E_\lambda$.

Future work:

- More TRS criteria for $E' \subseteq E_\lambda$ termination.
- Extend correspondence further:
  - *weak composite theories?*
  - *multi-sorted distributive laws?*

$\Longrightarrow$ proof

Lemma

Every $\mathbb{U}^\lambda$-term can be separated.

Proof.

Induction:

- Variables are separated ✓
- Take $\mathrm{op}(u_1, \ldots, u_n)$.
  - If $\mathrm{op} \in \Sigma^{\mathbb{T}}$:

$$\begin{aligned} \mathrm{op}(u_1, \ldots, u_n) &\overset{\text{IH}}{=}_{\mathbb{U}^\lambda} \mathrm{op}(t_1[s_x], \ldots, t_n[s_x]) \\ &= \mathrm{op}(t_1, \ldots, t_n)\,[s_x/x]\,. \end{aligned}$$

  - If $\mathrm{op} \in \Sigma^{\mathbb{S}}$,

$$\begin{aligned} \mathrm{op}(u_1, \ldots, u_n) &\overset{\text{IH}}{=}_{\mathbb{U}^\lambda} \mathrm{op}(t_1[s_x], \ldots, t_n[s_x]) \\ &= \mathrm{op}(t_1, \ldots, t_n)\,[s_x/x] \\ &\overset{\text{separation}}{=}_{\mathbb{U}^\lambda} t[s_y/y][s_x/x]. \end{aligned}$$

□

Attempt of proof without FRS

First attempt.

- $\mathbb{U}^\lambda$-terms can be separated: induction on terms.
- $t[s_x], t'[s'_x]$ $\mathbb{U}^\lambda$-equal $\implies$ equal modulo $(\mathbb{S}, \mathbb{T})$: induction on proof-tree
  - ...
  - Transitivity: $$\dfrac{\dfrac{\vdots}{t_1[s_1] = u} \qquad \dfrac{\vdots}{u = t_2[s_2]}}{t_1[s_1] = t_2[s_2]}$$

    Idea: separate at every level
    Problem: u can be separated differently

$\square$