Obtaining new quasitoposes:
Fuzzy presheaves and partially simple graphs

Aloïs Rosset
joint work with Roy Overbeek, Jörg Endrullis
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VRIJE
UNIVERSITEIT
AMSTERDAM

Rewriting

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String rewriting

$ab \to ba$

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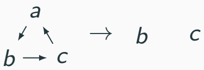
$ab \to ba$	$f(x, g(y)) \to f(x, x)$	

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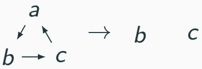
$ab \to ba$	$f(x, g(y)) \to f(x, x)$	

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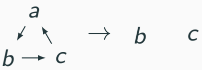
$ab \to ba$	$f(x, g(y)) \to f(x, x)$	

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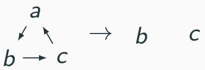
$ab \to ba$	$f(x, g(y)) \to f(x, x)$	

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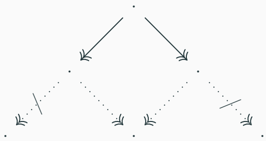

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- In rm-adhesive quasitoposes: Termination technique (weighted subgraph counting)

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Definition

Presheaf on category I is a functor $A : I^{\mathrm{op}} \to \mathrm{Set}$.

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$\begin{tikzcd} E \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & V \arrow[l, "\mathsf{refl}"'] \end{tikzcd}$ $s \cdot \mathsf{refl} = t \cdot \mathsf{refl} = \mathrm{id}_V$	Reflexive graphs (or *degenerate* graphs)

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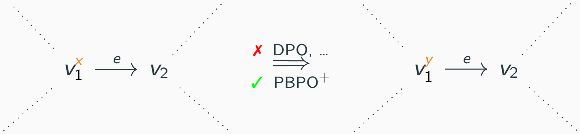

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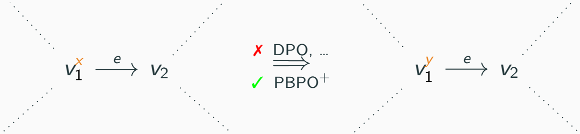

$v_1^{x} \xrightarrow{e} v_2$

✗ DPO, ...

$\Longrightarrow$

✓ PBPO$^+$

$v_1^{y} \xrightarrow{e} v_2$

Solution: label set $(\mathcal{L}, \leqslant)$ is complete lattice (join $\vee$, meet $\wedge$, bot $\bot$, top $\top$).

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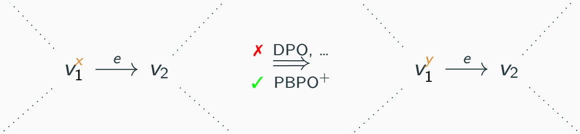

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$\Longrightarrow$ Similar to *fuzziness*.

Fuzziness

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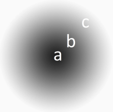

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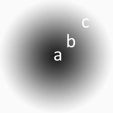

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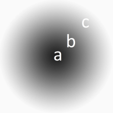

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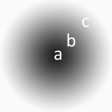

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Here, we see membership values as labels $v_1^x \xrightarrow{e^y} v_2^z$.

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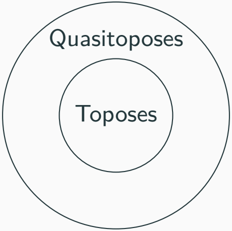

Presheaves properties

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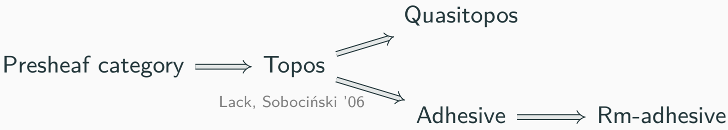

Presheaf category ⟹ Topos

Topos ⟹ Quasitopos

Topos ⟹ Adhesive

Lack, Sobociński '06

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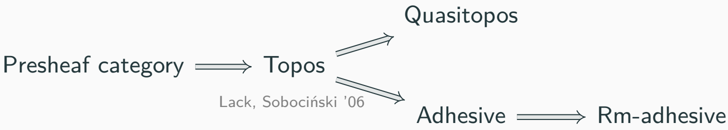

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Theorem (Stout'93)

$\mathcal{L}$ *Complete Heyting algebra* $\implies$ FuzzySet($\mathcal{L}$) *quasitopos.*

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$$\begin{tikzcd}
 & & \text{Quasitopos} & \\
\text{Presheaf category} \arrow[r, Rightarrow] & \text{Topos} \arrow[ru, Rightarrow] \arrow[rd, Rightarrow, "\text{Lack, Sobociński '06}"'] & & \\
 & & \text{Adhesive} \arrow[r, Rightarrow] & \text{Rm-adhesive}
\end{tikzcd}$$

Theorem (Stout'93)

$\mathcal{L}$ Complete Heyting algebra $\implies$ $\mathrm{FuzzySet}(\mathcal{L})$ quasitopos.

Theorem

Let I be a small category. $\mathcal{L} : I^{\mathrm{op}} \to \mathrm{CompHeytAlg}$:

$$\begin{tikzcd}
 & \textit{Quasitopos} \\
\mathcal{L}\textit{-Fuzzy Presheaf category} \arrow[ru, Rightarrow] \arrow[rd, Rightarrow] & \\
 & \textit{Rm-adhesive}
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Proof

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Lemma

Fuzzy presheaf: $\mathbf{1}(i) := \{\ \cdot^1\}$, $\forall i \in I$, *is a terminal object.*

Product

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Product of objects A, B is an object $A \times B$ with two projections $\pi_1 : A \times B \to A$, and $\pi_2 : A \times B \to B$, and a universal property.

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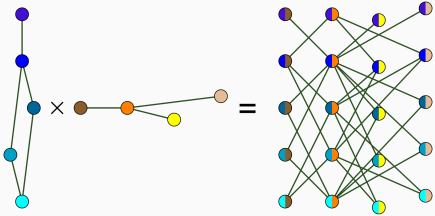

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Lemma

Fuzzy presheaves $(A, \alpha), (B, \beta) \implies (A \times B, \gamma)$*.*

$$\gamma_i((a, b) \in A(i) \times B(i)) = \alpha_i(a) \wedge \beta_i(b) \in \mathcal{L}(i).$$

Subobject classifier

Definition

Subobject classifier is $\mathrm{True} : 1 \to \Omega$ with

$$A \subseteq B \quad \Longleftrightarrow \quad \chi_A : B \to \Omega$$

subobjects characteristic function

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A \arrow[r, "!"] \arrow[d, tail, "m"'] \arrow[dr, phantom, "\lrcorner", very near start] & 1 \arrow[d, "\mathrm{True}"] \\
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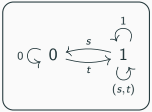
- Graph : $\Omega :=$ 

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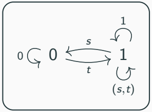

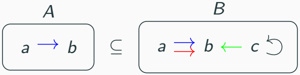

$$\Longleftrightarrow \chi_A : \begin{cases} a, b & \mapsto 1 \\ c & \mapsto 0 \\ {\color{blue}\rightarrow} & \mapsto \xrightarrow{1} \\ {\color{red}\rightarrow} & \mapsto \xrightarrow{(s,t)} \\ {\color{green}\leftarrow} & \mapsto \xrightarrow{t} \\ \circlearrowleft & \mapsto \xrightarrow{0} \end{cases}$$

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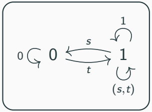
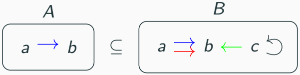
- Graph : $\Omega :=$ 



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- Presheaves: there is an Ω, generalising sets and graphs.

Regular-subobject classifier

- FuzzySet($\mathcal{L}$): ✗ **However**, $\Omega := \{0^1, 1^1\}$ classifies regular fuzzy subsets.

$$\underset{\text{is }\textbf{regular}}{\{a^x\} \subseteq \{a^{0.4}, b^{0.6}\}} \quad \Longleftrightarrow \quad x = 0.4$$

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Lemma

Regular-subobject classifier fuzzy presheaves:

Ω of presheaves + all elements full membership.

Cartesian closed

Definition

For objects A, B of a category, an **exponential object** is an object B^A s.t.

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Lemma

Fuzzy presheaves have exponential objects.

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Cartesian closedness stable under categorical equivalence. Hence:

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FuzzyPresheaf($I, \mathcal{L}$) *is locally cartesian closed because*

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Adhesivity

Definition (Partial)

Category is

- **adhesive**: if m mono, bottom PO, back PBs:

  front PBs $\iff$ top PO

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LT-Topologies

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Toposes have an internal logic (Goldblatt'06, McLarty'92)

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(for object X, then $\mathsf{True}_X := X \xrightarrow{!} 1 \xrightarrow{\mathsf{True}} \Omega$ means true *in context* X)

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LT-topology / **operator** / **modality** talks about “locally truth”.

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Induces ***closure*** operator on subobjects: $\overline{A_0 \rightarrowtail A}$: $\chi_{\overline{A_0 \rightarrowtail A}} := j \circ \chi_{A_0 \rightarrowtail A}$.

Subobject $A_0 \rightarrowtail A$ is **dense** if $\overline{A_0} = A$.

Examples of topologies

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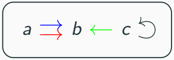
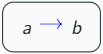
Illustration of $\neg\neg$-**topology** on Graph: for

A



A_0



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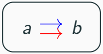

All topologies on $\mathsf{FuzzySet}(\mathcal{L})$

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Lemma

Similarly, topologies on FuzzyGraph($\mathcal{L}$) *are a combinations of topologies on* Graph *and of functions* $\phi_V, \phi_E : \mathcal{L} \to \mathcal{L}$.

Sep. & Sheaves

Separated elements

Standard topology: separated (Hausdorff) space B has property that any $f\colon A \to B$ is fully determined by its image on any dense subsets of A.

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Definition

Topology j, object B, arbitrary dense $m : A_0 \rightarrowtail A$ and arbitrary $f : A_0 \to B$. Count the number of factorisations $g : A \to B$ of f through m ($\forall A_0, A, f$):

- B is ***separated*** if $\#g \leqslant 1$,
- B is ***complete*** if $\#g \geqslant 1$,
- B is a ***sheaf*** if $\#g = 1$.

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A_0 \arrow[d, tail, "m"'] \arrow[dr, "f"] & \\
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Lemma (Johnstone '79)

For a topology, separated elements and sheaves form two quasitoposes.

(Un)directed simple graphs via topologies

Lemma (Vigna '03)

Simple graphs are the $\neg\neg$*-separated elements and form thus a quasitopos.*

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Proof.

Subgraph $A_0 \subseteq A$ is ¬¬-dense if A_0 has all vertices.

Then $\#g = \#$ way to maps edges of A onto edges of B.

This is $\leqslant 1$ iff B has at most one edge between each pair of vertices.

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Separated elements and sheaves in Set, Graph, FuzzyGraph($\mathcal{L}$)

Top.	Closure $\overline{A_0 \subseteq A}$	Dense object	Sep. elem.	Sheaves
Triv.	(adds everything)	all	subterminal objects	terminal objects
Dis.	(adds nothing)	only A	(all)	all
Closed	(adds all vertices)	if $A_0(E) = A(E)$	$\varnothing$, 	
$\neg\neg$	(adds all valid edges)	if $A_0(V) = A(V)$	simple graphs	complete graphs

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Similarly, simple fuzzy graphs are $\neg\neg$-sep. el. on FuzzyGraph($\mathcal{L}$).

Lemma

Simple fuzzy graphs form a quasitopos.

BiColGraph

$$\mathrm{Graph} = \mathrm{Set}^{I^{\mathrm{op}}} \text{ for } I^{\mathrm{op}} = \begin{tikzcd} E \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V \end{tikzcd} .$$

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$$\mathrm{BiColGraph} := \mathrm{Set}^{I^{\mathrm{op}}} \text{ for } I^{\mathrm{op}} = \begin{tikzcd} {\color{blue}E} \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & V & {\color{red}E'} \arrow[l, shift right, "s'"'] \arrow[l, shift left, "t'"] \end{tikzcd} . \; ({\color{blue}\rightarrow} \text{ and } {\color{red}\rightarrow})$$

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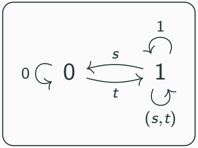
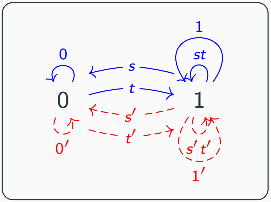
Ω_{Graph}



$\Omega_{\mathrm{BiColGraph}}$



Topologies on BiColGraph

Lemma

4 *topologies on* Graph $\Longrightarrow$ 8 *topologies on* BiColGraph.

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4 *topologies on* Graph $\implies$ 8 *topologies on* BiColGraph*.*

	j_1	j_2	j_3	j_4	j_5	j_6	j_7	j_8
on E	*disc.*	$\neg\neg$	*disc.*	$\neg\neg$	*closed*	*closed*	*triv.*	*triv.*
on E'	*disc.*	*disc.*	$\neg\neg$	$\neg\neg$	*closed*	*triv.*	*closed*	*triv.*

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$$\left.\begin{array}{l} j_2\text{-separated el.} = \text{blue-simple graphs.} \\ j_3\text{-separated el.} = \text{red-simple graphs.} \end{array}\right\} \implies \text{Partially simple graphs.}$$

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Corollary

Partially simple graphs form a quasitopos.

Separated elements in $\mathsf{FuzzySet}(\mathcal{L})$

Lemma

For a topology on $\mathsf{FuzzySet}(\mathcal{L})$ *corresponding to some* $\phi : \mathcal{L} \to \mathcal{L}$*:*

- *every fuzzy set is separated,*
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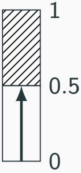
Example: $\mathcal{L} \coloneqq [0,1]$, $\phi(x) \coloneqq \max\{x, 0.5\} = x \vee 0.5$
A fuzzy set is a sheaf if its membership are in $[0.5, 1]$.



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(Quasi)topos	Graph	BiColGraph	Fuzzy graphs
($\neg\neg$-separated elements) Quasitopos	Simple graphs	Partially simple graphs	Simple fuzzy graphs

Future work

Future work:

- Define “fuzzy quasitopos”? Are they quasitoposes?
- Obtain more quasitoposes via topologies.
- Specifically, looking at other presheaf categories (they are toposes).
- Can we deduce the internal logic of the quasitopos obtained from the internal logic of the topos?

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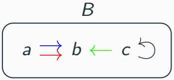

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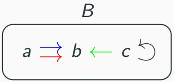

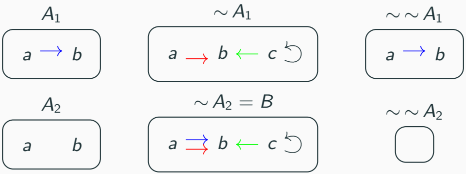

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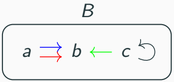

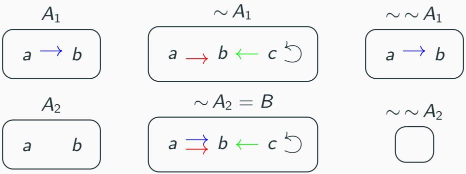

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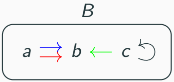

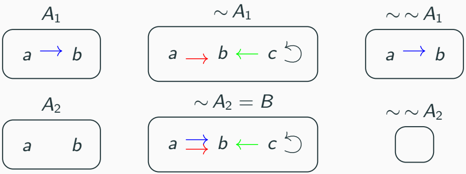

Conegation seem to be more of an *interior* operation.

Questions: Does it capture local falsehood? Does it satisfy the dual axioms? Can we obtain other new quasitoposes?