Fuzzy Presheaves are Quasitoposes

Aloïs Rosset
joint work with Roy Overbeek and Jörg Endrullis

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Vrije Universiteit Amsterdam

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Why fuzzy presheaves? Practical graph relabelling.

Graphs as Presheaves

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Let I^{op} be a category.

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$\Rightarrow$ category $\mathrm{Set}^{I^{\mathrm{op}}}$ or $\mathrm{Presheaf}(I)$.

Examples

The category I^{op} is a template.

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Category I^{op}	Presheaf A	Category Presheaf(I)
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$\begin{tikzcd} E \arrow[r, shift left=2, "s"] \arrow[r, shift right=2, "t"'] & V \arrow[l, "\mathrm{refl}"'] \end{tikzcd}$ $s \cdot \mathrm{refl} = t \cdot \mathrm{refl} = \mathrm{id}_V$	specific loop $\begin{tikzcd} v \arrow[loop left, "\mathrm{refl}_v"] \end{tikzcd}$	Reflexive graph (or *degenerate* graphs)

Presheaves properties

Lemma

Let I be a small category.

$$\begin{tikzcd}
 & & \textit{Quasitopos} & \\
\textit{Presheaf category} \arrow[r, Rightarrow] & \textit{Topos} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow, "\textit{Lack, Sobociński '06}"'] & & \\
 & & \textit{Adhesive} \arrow[r, Rightarrow] & \textit{Rm-adhesive}
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Fuzziness

Fuzzy sets

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A **fuzzy set** (A, α) is a set A with a membership function $\alpha : A \to [0, 1]$.

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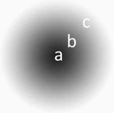
For instance: $A = \{a^1, b^{0.8}, c^{0.4}, \dots\}$



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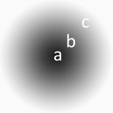

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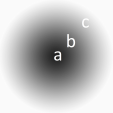

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More general definition: poset $(\mathcal{L}, \leqslant)$ instead of $[0, 1]$.

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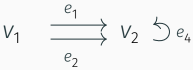

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For some authors: $v_1^x \xrightarrow{e^z} v_2^y \quad \Rightarrow \quad x, y \geqslant z.$

Why fuzzy structure?

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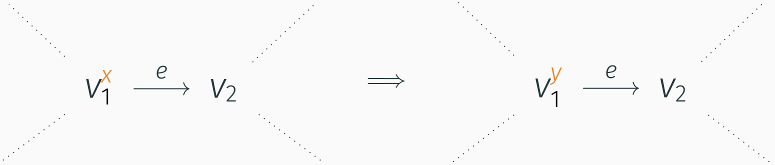
When $(\mathcal{L}, \leqslant)$ is a complete lattice (i.e., join $\vee$, meet $\wedge$, bot $\bot$, top $\top$), PBPO$^+$ offers an easy treatment of *graph relabelling*,



Fuzzy presheaves

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Given category I and functor $\mathcal{L} : I^{\mathrm{op}} \to \mathrm{Poset}$, a (poset) ***$\mathcal{L}$-fuzzy presheaf*** is a presheaf $A : I^{\mathrm{op}} \to \mathrm{Set}$ and a *lax natural transformation* $\alpha : \mathrm{incl} \cdot A \Rightarrow \mathcal{L}$.

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A(i) \arrow[r, "A(\iota)"] \arrow[d, "\alpha_i"'] \arrow[dr, phantom, "\leqslant"] & A(j) \arrow[d, "\alpha_j"] \\
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- $I^{\mathrm{op}} =$ $\begin{tikzcd} E \arrow[r, shift left=3, "s_1"] \arrow[r, "s_2"] \arrow[r, shift right=3, "s_3"'] & V \end{tikzcd}$ ⇒ fuzzy 3-uniform hypergraphs
- ...

Quasitopos

Theorem (Stout'93)

$\mathcal{L}$ Complete Heyting algebra $\implies$ $\mathrm{FuzzySet}(\mathcal{L})$ quasitopos.

Theorem

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 & \textit{Rm-adhesive}
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Quasitopos

Quasitopos

To be a quasitopos, three things needed:

- finite (co)limits
    - terminal object, products, equalisers.
    - initial object, coproduct, coequalisers.
- regular-subobject classifier
- locally cartesian closed

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Lemma

Fuzzy presheaf: $\mathbf{1}(i) \coloneqq \{\ \cdot^1\}, \forall i \in I$, *is a terminal object.*

Product

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- Sets $A, B \implies A \times B$

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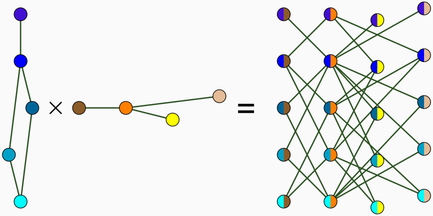

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Lemma

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$$\gamma_i((a, b) \in A(i) \times B(i)) = \alpha_i(a) \wedge \beta_i(b) \in \mathcal{L}(i).$$

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Definition (Informally)

Subobject classifier is an object Ω s.t.

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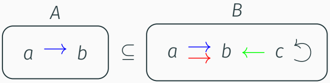
- Subgraph 

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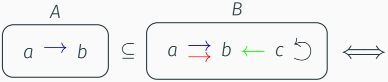

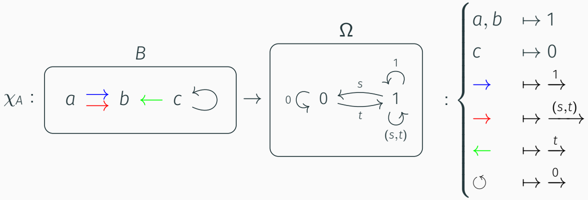

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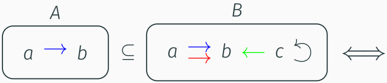
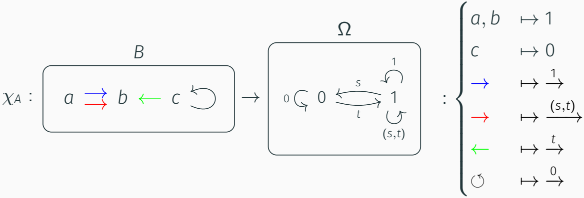
- Subgraph 



- Presheaves: there is an Ω, generalising sets and graphs.

Regular-subobject classifier

- FuzzySet($\mathcal{L}$): **no** Ω for fuzzy subsets
  However, $\Omega := \{0^1, 1^1\}$ classifies regular fuzzy subsets.
  $\{a^x\} \subseteq \{a^{0.4}, b^{0.6}\}$ is regular iff $x = 0.4$.

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Lemma

Regular-subobject classifier fuzzy presheaves:

$$\textit{presheaves } \Omega + \textit{all elements full membership.}$$

Cartesian closed

Definition

For objects A, B of a category, an **exponential object** is an object B^A s.t.

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Fuzzy presheaves have exponential objects.

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Cartesian closedness stable under categorical equivalence. Hence:

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Lemma (e.g., Awodey'05)

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Lemma

FuzzyPresheaf($I, \mathcal{L}$) *is locally cartesian closed because*

$$\text{FuzzyPresheaf}(I, \mathcal{L})\big/_{(D, \delta)} \quad \simeq \quad \text{FuzzyPresheaf}(\mathrm{el}(D, \delta), \tilde{\mathcal{L}})$$

Rm-adhesivity

Adhesivity

Definition (Partial)

Category is

- **adhesive**: if m mono, bottom PO, back PBs:

  front PBs $\iff$ top PO
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$$\begin{tikzcd}
 & F \arrow[rr] \arrow[dl] \arrow[dd] & & G \arrow[dl] \arrow[dd] \\
E \arrow[rr, crossing over] \arrow[dd] & & H & \\
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Theorem

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Topos \arrow[r, Rightarrow, "\text{Lack, Sobociński '06}"] \arrow[dd, Rightarrow] & Adhesive \arrow[d, Rightarrow] \\
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\mathit{Quasitopos} \arrow[urr, Rightarrow, "\text{if regular subobjects closed under binary union Johnstone et al. '07}"] \arrow[rr, Rightarrow, "\text{Garner, Lack '12}"'] & & \mathit{Rm-quasiadhesive}
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Simple graphs

Simple graphs via topologies

Simple graphs do not form a presheaf category, however

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The proof uses a notion of **topology** and of **separated elements**.

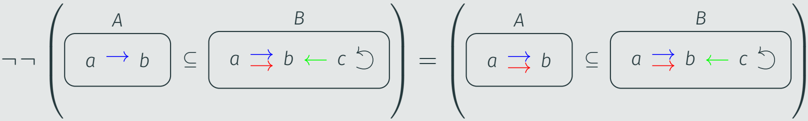
Lemma (Johnstone ’79)

In a topos, separated elements for a topology form a quasitopos.

Separated elements

Definition (Informally)

In Graph, the **double negation topology** can be illustrated by



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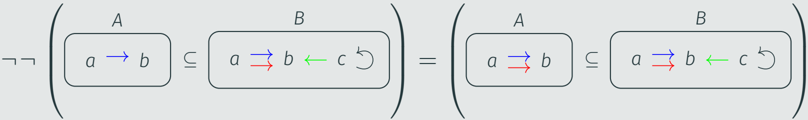

(Quasi)Topos	Graphs	Fuzzy graphs
Quasitopos (¬¬-separated elements)	Simple graphs	Simple fuzzy graphs

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Future work:

- Define “fuzzy quasitopos”? Are they quasitoposes?
- Obtain more quasitoposes via topologies.