Fuzzy Presheaves are Quasitoposes

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joint work with Roy Overbeek and Jörg Endrullis

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Vrije Universiteit Amsterdam

Fuzzy sets

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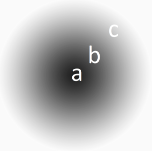

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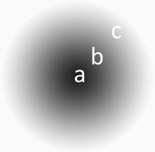

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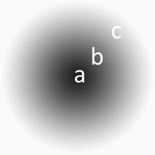

More general definition:
poset $(\mathcal{L}, \leqslant)$ instead of $[0, 1]$.

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For some authors: $v_1^x \xrightarrow{e^z} v_2^y \quad \Rightarrow \quad x, y \geqslant z.$

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- composition of arrows $a \xrightarrow{g \circ f} b, \quad \ldots$

$$\begin{tikzcd}
a \arrow[loop above, "\mathrm{id}_a"] \arrow[r, "f"] \arrow[rr, bend right=15, "h"] \arrow[rr, bend right=35, "g \circ f"'] & b \arrow[loop above, "\mathrm{id}_b"] \arrow[r, "g"] & c \arrow[loop above, "\mathrm{id}_c"]
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- VectSpace: category of *vector spaces* and *linear transformations*

Graphs as Presheaves

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$\Longrightarrow$ A is a graph.

More examples

- $I^{\mathrm{op}} = \quad E \overset{\xrightarrow{s_1}}{\underset{\xrightarrow{s_3}}{\xrightarrow{s_2}}} V$ gives $\overbrace{e = (v_1, v_2, v_3)}^{\text{hyperedge size 3}}$

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- $I^{\mathrm{op}} = E \; \begin{matrix} \xrightarrow{s_1} \\ \cdots \\ \xrightarrow{s_k} \end{matrix} \; V$ gives $\overbrace{e = (v_1, \ldots, v_k)}^{\text{hyperedge size } k} \implies$ k-uniform hypergraphs
- $I^{\mathrm{op}} =$ $$\begin{tikzcd} E_2 \arrow[dr, shift left] \arrow[dr, shift right] & \\ E_3 \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & V \\ \ldots & \end{tikzcd}$$ $\implies$ hypergraphs

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- $I^{\mathrm{op}} = E \xleftarrow{f} R \xrightarrow{g} V$. Given a presheaf G, read $r \in A(R)$ as meaning

  "vertex $A(g)(r) \in A(V)$ is incident to edge $A(f)(r) \in A(E)$".

  $\Rightarrow$ multiple incidences hypergraphs

More examples (continued)

Let us come back to 2-element edges.

- Undirected graphs: $e = (v_1, v_2),\ \mathsf{sym}(e) = (v_2, v_1) \iff e = \{v_1, v_2\}$

$$I^{\mathrm{op}} = \ \mathsf{sym} \circlearrowleft E \overset{s}{\underset{t}{\rightrightarrows}} V \qquad \begin{cases} \mathsf{sym} \cdot \mathsf{sym} = \mathrm{id}_E, \\ s \cdot \mathsf{sym} = t, \\ t \cdot \mathsf{sym} = s. \end{cases}$$

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- Undirected reflexive: Both sym and refl. Extra equation: $\mathsf{sym} \cdot \mathsf{refl} = \mathsf{refl}$.

Graphs homomorphisms as natural transformations

What about graph homomorphism?

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Presheaves $A, B : I^{\mathrm{op}} \to \mathrm{Set}$. A **natural transformation** $f : A \Rightarrow B$ is

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  Thus, $\mathrm{Presheaf}(I) \cong \mathrm{Set}$.
- For $I^{\mathrm{op}} = E \underset{t}{\overset{s}{\rightrightarrows}} V$, then $f : A \Rightarrow B$ is a *graph homomorphism*:
  - two function $f_E : A(E) \to B(E)$ and $f_V : A(V) \to B(V)$
  - such that $B(s) \circ f_E = f_V \circ A(s)$ and $B(t) \circ f_E = f_V \circ A(t)$

  Thus $\mathrm{Presheaf}(I) \cong \mathrm{Graph}$.

Presheaves properties

$$
\begin{tikzcd}
 & & \text{Quasitopos} & \\
\text{Presheaf category} \arrow[r, Rightarrow] & \text{Topos} \arrow[ur, Rightarrow] \arrow[dr, Rightarrow] & & \\
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Recall (last TCS seminar) [Overbeek et al, 2023]:

- in quasitopos: PBPO^+ subsumes DPO, SqPO, AGREE, PBPO
- in rm-adhesive quasitopos: termination method using weighted subgraph counting.

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Membership values can be seen as labels $v_1 \xrightarrow{e^x} v_2$.

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When $(\mathcal{L}, \leqslant)$ is a complete lattice (i.e., join $\vee$, meet $\wedge$, bot $\perp$, top $\top$), PBPO$^+$ offers an easy treatment of *graph relabelling*,
i.e., changing a label x (of an edge or vertex) to another label y

Presentations of monads by algebraic theories

Definition (Informally)

A **fuzzy presheaf** is a presheaf $A : I^{\mathrm{op}} \to \mathrm{Set}$ where every set $A(i)$ has a membership function $\alpha_i : A(i) \to \mathcal{L}(i)$ to some poset $(\mathcal{L}(i), \leqslant)$.

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(A **fuzzy presheaf morphism** is a natural transformation $f : (A, \alpha) \to (B, \beta)$ such that $\alpha_i \leqslant \beta_i f_i$ for all $i \in I^{\mathrm{op}}$.)

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Quasitopos

Theorem

If I is a small category and if every $\mathcal{L}(i)$ is a complete Heyting algebra, then

$$\begin{tikzcd} & \textit{Quasitopos} \\ \textit{Fuzzy Presheaves category} \arrow[ur, Rightarrow, "\textit{(proof now)}"] \arrow[dr, Rightarrow, "\textit{(out of scope)}"'] & \\ & \textit{Rm-adhesive} \end{tikzcd}$$

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To be a quasitopos, three things needed:

- finite limits, finite colimits
- regular-subobject classifier
- exponent objects (roughly)

Finite (co)limits

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Lemma

Fuzzy presheaf $1(i) := \{\ \cdot^1\}$ *for all* $i \in I$ *is a terminal object.*

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Definition (Informally)

In a category, the **product** of objects A, B is an object $A \times B$ with two projections $\pi_1 : A \times B \to A$, and $\pi_2 : A \times B \to B$, and ...

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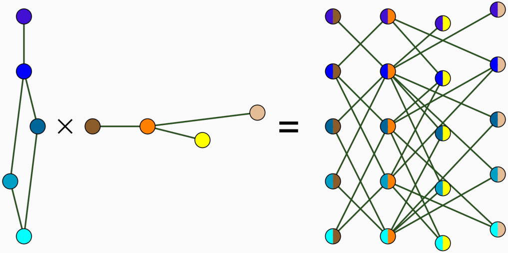

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Lemma

Fuzzy presheaves $(A, \alpha), (B, \beta) \implies$ *fuzzy presheaf* $(A \times B, \gamma)$.
Membership: $\gamma_i((a, b) \in A(i) \times B(i)) = \alpha_i(a) \wedge \beta_i(b)$ *in* $\mathcal{L}(i)$.

Last limits

There are other limits to verify: equalizers or pullbacks, but it is similar.

Finite colimits are dual:

- initial object
- coproduct (i.e., disjoint union)

Classifier Ω

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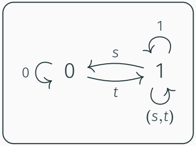

$$: \begin{cases} a, b & \mapsto 1 \\ c & \mapsto 0 \\ \to & \mapsto \xrightarrow{1} \\ \to & \mapsto \xrightarrow{(s,t)} \\ \leftarrow & \mapsto \xrightarrow{t} \\ \circlearrowleft & \mapsto \xrightarrow{0} \end{cases}$$

Regular-subobject classifier

- Let $\Omega \coloneqq \{0^1, 1^1\}$. Fuzzy subsets $A_1 = \{a^{0.1}\}, A_2 = \{a^{0.4}\} \subseteq B = \{a^{0.4}, b^{0.6}\}$.

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Lemma

The subobject classifier Ω for presheaves, with every element given full membership 1 is a regular-subobject classifier for fuzzy presheaves.

Exponential

Definition

For objects A, B of a category, an **exponential object** is an object B^A s.t.

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Future work: Define "fuzzy quasitopos"? Prove they are quasitoposes?