Algebraic Presentation of Semifree Monads

Aloïs Rosset
Joint work with Helle Hvid Hansen and Jörg Endrullis

November 29, 2021

Vrije Universiteit Amsterdam

Motivation

Distributive laws of T over M	$\overset{\text{Beck [2]}}{\Longleftrightarrow}$	Liftings $\tilde{T}$ on M-algebras

Motivation

Distributive laws of T over M	$\overset{\text{Beck [2]}}{\Longleftrightarrow}$	Liftings $\tilde{T}$ on M-algebras
Weak distributive laws of T over M	$\overset{\text{Garner [3]}}{\Longleftrightarrow}$	Subclass of liftings $\tilde{T}$ on M-semialgebras

Motivation

Distributive laws of T over M $\overset{\text{Beck [2]}}{\Longleftrightarrow}$ Liftings $\tilde{T}$ on M-algebras

Weak distributive laws of T over M $\overset{\text{Garner [3]}}{\Longleftrightarrow}$ Subclass of liftings $\tilde{T}$ on M-semialgebras

M-semialgebras
$= M^{\mathrm{s}}$-algebras
Petrișan & Sarkis [12]

Motivation

Distributive laws of T over M $\overset{\text{Beck [2]}}{\Longleftrightarrow}$ Liftings $\tilde{T}$ on M-algebras

Weak distributive laws of T over M $\overset{\text{Garner [3]}}{\Longleftrightarrow}$ Subclass of liftings $\tilde{T}$ on M-semialgebras

M-semialgebras
$= M^{\mathrm{s}}$-algebras
Petrișan & Sarkis [12]

$\Updownarrow$

Subclass of liftings $\tilde{T}$ on M^{s}-algebras

Motivation

Distributive laws of T over M	$\overset{\text{Beck [2]}}{\underset{(*)}{\Longleftrightarrow}}$	Liftings $\tilde{T}$ on M-algebras
Weak distributive laws of T over M	$\overset{\text{Garner [3]}}{\Longleftrightarrow}$	Subclass of liftings $\tilde{T}$ on M-semialgebras
	M-semialgebras $= M^{\mathrm{s}}$-algebras Petrișan & Sarkis [12]	$\Updownarrow$
Subclass of distrib. laws of T over M^{s}	$\underset{(*)}{\Longleftrightarrow}$	Subclass of liftings $\tilde{T}$ on M^{s}-algebras

Goal

Theorem
Let (M, η, μ) be a monad on Set*. If (Σ, E) is a concrete algebraic presentation of (M, η, μ) then $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ is a concrete algebraic presentation of the semifree monad $(M^{\mathrm{s}}, \eta^{\mathrm{s}}, \mu^{\mathrm{s}})$.*

Overview

- Monads and distributive laws
- Algebraic Theories
- Presentations of monads by algebraic theories
- The *semifree monad* M^{s} on a monad M
- The general presentation of M^{s}
- Examples

Monads

A *monad* [9] is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ with a *unit* $\eta_X : X \to MX$ and a *multiplication* $\mu_X : MMX \to X$

Monads

A *monad* [9] is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ with a *unit* $\eta_X : X \to MX$ and a *multiplication* $\mu_X : MMX \to X$ rendering commutative

$$
\begin{array}{ccc}
M & \xrightarrow{M\eta} & M^2 \\
{\scriptstyle \eta M}\downarrow & \searrow{\scriptstyle =} & \downarrow{\scriptstyle \mu} \\
M^2 & \xrightarrow[\mu]{} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
M^3 & \xrightarrow{\mu M} & M^2 \\
{\scriptstyle M\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
M^2 & \xrightarrow[\mu]{} & M
\end{array}
$$

Monads

A *monad* [9] is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ with a *unit* $\eta_X : X \to MX$ and a *multiplication* $\mu_X : MMX \to X$ rendering commutative

Examples in the category Set:

- The *maybe* monad $X \mapsto X + 1$. The η is the left injection $X \to X + 1$ and the $\mu_X : (X + 1) + 1 \to X + 1$ combines both "+1".

Monads

A *monad* [9] is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ with a *unit* $\eta_X : X \to MX$ and a *multiplication* $\mu_X : MMX \to X$ rendering commutative

$$\begin{array}{ccc} M & \xrightarrow{M\eta} & M^2 \\ {\scriptstyle \eta M}\downarrow & \searrow\!\!\!\!= & \downarrow{\scriptstyle \mu} \\ M^2 & \xrightarrow[\mu]{} & M \end{array} \qquad\qquad \begin{array}{ccc} M^3 & \xrightarrow{\mu M} & M^2 \\ {\scriptstyle M\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\ M^2 & \xrightarrow[\mu]{} & M \end{array}$$

Examples in the category Set:

- The *maybe* monad $X \mapsto X + 1$. The η is the left injection $X \to X + 1$ and the $\mu_X : (X + 1) + 1 \to X + 1$ combines both "+1".
- The *powerset* monad $\mathcal{P}$ with $\{-\} : X \to \mathcal{P}X$ and $\bigcup : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$.

Monads

A *monad* [9] is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ with a *unit* $\eta_X : X \to MX$ and a *multiplication* $\mu_X : MMX \to X$ rendering commutative

$$
\begin{array}{ccc}
M & \xrightarrow{M\eta} & M^2 \\
{\scriptstyle \eta M}\downarrow & \diagdown\!\!\diagdown & \downarrow{\scriptstyle \mu} \\
M^2 & \xrightarrow[\mu]{} & M
\end{array}
\qquad\qquad
\begin{array}{ccc}
M^3 & \xrightarrow{\mu M} & M^2 \\
{\scriptstyle M\mu}\downarrow & & \downarrow{\scriptstyle \mu} \\
M^2 & \xrightarrow[\mu]{} & M
\end{array}
$$

Examples in the category Set:

- The *maybe* monad $X \mapsto X + 1$. The η is the left injection $X \to X + 1$ and the $\mu_X : (X + 1) + 1 \to X + 1$ combines both "+1".
- The *powerset* monad $\mathcal{P}$ with $\{-\} : X \to \mathcal{P}X$ and $\bigcup : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$.
- The *list* monad L with $[-] : X \to LX$ and concat. : $LLX \to LX$.

Monads

A *monad* [9] is an endofunctor $M : \mathsf{C} \to \mathsf{C}$ with a *unit* $\eta_X : X \to MX$ and a *multiplication* $\mu_X : MMX \to X$ rendering commutative

Examples in the category Set:

- The *maybe* monad $X \mapsto X + 1$. The η is the left injection $X \to X + 1$ and the $\mu_X : (X + 1) + 1 \to X + 1$ combines both "+1".
- The *powerset* monad $\mathcal{P}$ with $\{-\} : X \to \mathcal{P}X$ and $\bigcup : \mathcal{P}\mathcal{P}X \to \mathcal{P}X$.
- The *list* monad L with $[-] : X \to LX$ and concat. $: LLX \to LX$.
- The *distribution* monad $\mathcal{D}$ with dirac $\delta_X : X \to \mathcal{D}X$ and flattening $\mathrm{flat} : \mathcal{D}\mathcal{D}X \to \mathcal{D}X$.

Algebra for a monad

An *M-algebra* is a object X together with a morphism $\alpha : MX \to X$

Algebra for a monad

An *M-algebra* is a object X together with a morphism $\alpha : MX \to X$ rendering commutative

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & MX \\ & \searrow^{=} & \downarrow \alpha \\ & & X \end{array} \qquad (1) \qquad\qquad \begin{array}{ccc} M^2X & \xrightarrow{\mu_X} & MX \\ {\scriptstyle M\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ MX & \xrightarrow[\alpha]{} & X \end{array} \qquad (2)$$

Algebra for a monad

An *M-algebra* is a object X together with a morphism $\alpha : MX \to X$ rendering commutative

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & MX \\ & \searrow{=} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array} \qquad (1) \qquad\qquad \begin{array}{ccc} M^2X & \xrightarrow{\mu_X} & MX \\ {\scriptstyle M\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ MX & \xrightarrow[\alpha]{} & X \end{array} \qquad (2)$$

Together with the appropriate notion of morphism, they form the category $\mathbf{EM}(M)$.

Algebra for a monad

An *M-algebra* is a object X together with a morphism $\alpha : MX \to X$ rendering commutative

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & MX \\ & \searrow^{=} & \downarrow{\scriptstyle \alpha} \\ & & X \end{array} \qquad (1) \qquad\qquad \begin{array}{ccc} M^2X & \xrightarrow{\mu_X} & MX \\ {\scriptstyle M\alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ MX & \xrightarrow[\alpha]{} & X \end{array} \qquad (2)$$

Together with the appropriate notion of morphism, they form the category $\mathbf{EM}(M)$.

M-semialgebras are only required to satisfy (2) and form the category $\mathbf{EM}_s(M)$ (Garner,[3]).

Distributive law

Given two monads T and M, a *distributive law* [2] of T over M is a natural transformation $\lambda : MT \Rightarrow TM$ rendering (3)-(6) commutative.

Distributive law

Given two monads T and M, a *distributive law* [2] of T over M is a natural transformation $\lambda : MT \Rightarrow TM$ rendering (3)-(6) commutative.

$$\begin{array}{ccc} & T & \\ {\scriptstyle \eta^M{}_T}\swarrow & & \searrow{\scriptstyle T\eta^M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \qquad (3)$$

$$\begin{array}{ccc} & M & \\ {\scriptstyle M\eta^T}\swarrow & & \searrow{\scriptstyle \eta^T{}_M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \qquad (4)$$

$$\begin{array}{ccccc} MMT & \xrightarrow{M\lambda} & MTM & \xrightarrow{\lambda M} & TMM \\ {\scriptstyle \mu^M{}_T}\downarrow & & & & \downarrow{\scriptstyle T\mu^M} \\ MT & & \xrightarrow[\lambda]{} & & TM \end{array} \qquad (5)$$

$$\begin{array}{ccccc} MTT & \xrightarrow{\lambda T} & MTM & \xrightarrow{T\lambda} & TMM \\ {\scriptstyle M\mu^T}\downarrow & & & & \downarrow{\scriptstyle \mu^T{}_M} \\ MT & & \xrightarrow[\lambda]{} & & TM \end{array} \qquad (6)$$

Lemma:(Beck,[2]) There is a bijective correspondence between distributive laws $\lambda : MT \Rightarrow TM$ and monad structures on TM.

Distributive law

Given two monads T and M, a *distributive law* [2] of T over M is a natural transformation $\lambda : MT \Rightarrow TM$ rendering (3)-(6) commutative.

$$\begin{array}{ccc} & T & \\ {\scriptstyle \eta^M T}\swarrow & & \searrow{\scriptstyle T\eta^M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \qquad (3)$$

$$\begin{array}{ccc} & M & \\ {\scriptstyle M\eta^T}\swarrow & & \searrow{\scriptstyle \eta^T M} \\ MT & \xrightarrow[\lambda]{} & TM \end{array} \qquad (4)$$

$$\begin{array}{ccccc} MMT & \xrightarrow{M\lambda} & MTM & \xrightarrow{\lambda M} & TMM \\ {\scriptstyle \mu^M T}\downarrow & & & & \downarrow{\scriptstyle T\mu^M} \\ MT & & \xrightarrow[\lambda]{} & & TM \end{array} \qquad (5)$$

$$\begin{array}{ccccc} MTT & \xrightarrow{\lambda T} & MTM & \xrightarrow{T\lambda} & TMM \\ {\scriptstyle M\mu^T}\downarrow & & & & \downarrow{\scriptstyle \mu^T M} \\ MT & & \xrightarrow[\lambda]{} & & TM \end{array} \qquad (6)$$

A *weak distributive law* is only required to satisfy (4)-(6).

Lemma:(Beck,[2]) There is a bijective correspondence between distributive laws $\lambda : MT \Rightarrow TM$ and monad structures on TM.

Algebraic theories

An *algebraic theory* [10] is a pair (Σ, E), where

- Σ is a set of operations symbols $f, g, \ldots$ each with an arity $n \in \mathbb{N}$.
- E is a set of *equations* over Σ, i.e., pairs of terms (s, t) built using the symbols in Σ and a set of variables, say Var.

Algebraic theories

An *algebraic theory* [10] is a pair (Σ, E), where

- Σ is a set of operations symbols $f, g, \ldots$ each with an arity $n \in \mathbb{N}$.
- E is a set of *equations* over Σ, i.e., pairs of terms (s, t) built using the symbols in Σ and a set of variables, say Var.

Examples:

- Theory of *pointed sets* $\Sigma := \{\bullet^{(0)}\}$, $E := \varnothing$.

Algebraic theories

An *algebraic theory* [10] is a pair (Σ, E), where

- Σ is a set of operations symbols $f, g, \ldots$ each with an arity $n \in \mathbb{N}$.
- E is a set of *equations* over Σ, i.e., pairs of terms (s, t) built using the symbols in Σ and a set of variables, say Var.

Examples:

- Theory of *pointed sets* $\Sigma := \{\bullet^{(0)}\}$, $E := \varnothing$.
- Theory of *monoids* $\Sigma := \{e^{(0)}, \cdot^{(2)}\} E := \left\{ \begin{array}{c} v \cdot e = v, \\ e \cdot v = v, \\ (u \cdot v) \cdot w = u \cdot (v \cdot w) \end{array} \right\}$

Algebraic theories

An *algebraic theory* [10] is a pair (Σ, E), where

- Σ is a set of operations symbols $f, g, \ldots$ each with an arity $n \in \mathbb{N}$.
- E is a set of *equations* over Σ, i.e., pairs of terms (s, t) built using the symbols in Σ and a set of variables, say Var.

Examples:

- Theory of *pointed sets* $\Sigma := \{\bullet^{(0)}\}$, $E := \varnothing$.
- Theory of *monoids* $\Sigma := \{e^{(0)}, \cdot^{(2)}\} E := \left\{ \begin{array}{c} v \cdot e = v, \\ e \cdot v = v, \\ (u \cdot v) \cdot w = u \cdot (v \cdot w) \end{array} \right\}$
- Theory of *convex sets* $\Sigma := \{+_p : 2 \mid p \in (0,1)\}$,

$$E := \left\{ \begin{array}{c} v +_p v = v, \\ u +_p v = v +_{1-p} u, \\ (u +_q v) +_p w = u +_{pq} (v +_{p(1-q)/1-pq} w) \end{array} \right\}.$$

Algebra for an algebraic theory

Given (Σ, E), a *(Σ, E)-algebra* is a set X, and a collection of interpretations $[\![f]\!] : X^n \to X$ for each n-ary $f \in \Sigma$, such that the equations are satisfied.

Algebra for an algebraic theory

Given (Σ, E), a *(Σ, E)-algebra* is a set X, and a collection of interpretations $[\![f]\!] : X^n \to X$ for each n-ary $f \in \Sigma$, such that the equations are satisfied.

An equation (s, t) is *satisfied* if $[\![s]\!]_\sigma = [\![t]\!]_\sigma$ for all variable assignment $\sigma : \mathsf{Var} \to X$, where $[\![-]\!]_\sigma$ is the unique extension of all interpretations $[\![f]\!]$ on the terms

$$[\![-]\!]_\sigma : \mathcal{T}_\Sigma(\mathsf{Var}) \to X.$$

Algebra for an algebraic theory

Given (Σ, E), a *(Σ, E)-algebra* is a set X, and a collection of interpretations $[\![f]\!] : X^n \to X$ for each n-ary $f \in \Sigma$, such that the equations are satisfied.

An equation (s, t) is *satisfied* if $[\![s]\!]_\sigma = [\![t]\!]_\sigma$ for all variable assignment $\sigma : \mathsf{Var} \to X$, where $[\![-]\!]_\sigma$ is the unique extension of all interpretations $[\![f]\!]$ on the terms

$$[\![-]\!]_\sigma : \mathcal{T}_\Sigma(\mathsf{Var}) \to X.$$

Together with the appropriate notion of morphism, they form the category $\mathbf{Alg}(\Sigma, E)$.

Presentations of monads by algebraic theories

An algebraic theory (Σ, E) is an *algebraic presentation* of a monad M if their categories of algebras are isomorphic: $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma, E)$.

Presentations of monads by algebraic theories

An algebraic theory (Σ, E) is an *algebraic presentation* of a monad M if their categories of algebras are isomorphic: $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma, E)$.

We have a *concrete algebraic presentation* when the isomorphic is concrete.

Presentations of monads by algebraic theories

An algebraic theory (Σ, E) is an *algebraic presentation* of a monad M if their categories of algebras are isomorphic: $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma, E)$.

We have a *concrete algebraic presentation* when the isomorphic is concrete.

Examples:

- The theory of pointed sets presents the maybe monad, i.e., an algebra $X + 1 \to X$ corresponds to an algebra $(X, [\![\bullet]\!] \in X)$.

Presentations of monads by algebraic theories

An algebraic theory (Σ, E) is an *algebraic presentation* of a monad M if their categories of algebras are isomorphic: $\mathbf{EM}(M) \cong \mathbf{Alg}(\Sigma, E)$.

We have a *concrete algebraic presentation* when the isomorphic is concrete.

Examples:

- The theory of pointed sets presents the maybe monad, i.e., an algebra $X + 1 \to X$ corresponds to an algebra $(X, [\![\bullet]\!] \in X)$.
- The theory of monoids presents the list monad. [1, Ex. 10.7]
- The theory of convex sets presents the distribution monad. [6]

Semifree monads

Given a monad M, the authors in

Daniela Petrian and Ralph Sarkis. "Semialgebras and Weak Distributive Laws". In: *Proceedings MFPS 2021* ().

recently introduced the *semifree monad* M^{s},

Semifree monads

Given a monad M, the authors in

Daniela Petrian and Ralph Sarkis. "Semialgebras and Weak Distributive Laws". In: *Proceedings MFPS 2021* ().

recently introduced the *semifree monad* M^{s}, with properties

- $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$

Semifree monads

Given a monad M, the authors in

Daniela Petrian and Ralph Sarkis. "Semialgebras and Weak Distributive Laws". In: *Proceedings MFPS 2021* ().

recently introduced the *semifree monad M^{s}*, with properties

- $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$ (M^{s}-algebras $\cong$ M-semialgebras)

Semifree monads

Given a monad M, the authors in

Daniela Petrian and Ralph Sarkis. "Semialgebras and Weak Distributive Laws". In: *Proceedings MFPS 2021* ().

recently introduced the *semifree monad* M^{s}, with properties

- $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$ $\quad$ (M^{s}-algebras $\cong$ M-semialgebras)
- $\{\lambda : MT \overset{\text{w.d.l}}{\Rightarrow} TM\} \overset{\text{bijection}}{\leftrightarrow} \{\delta : M^{\mathrm{s}}T \overset{\text{d.l}}{\Rightarrow} TM^{\mathrm{s}} \text{ satisfying an extra condition}\}$.

Semifree monads

Given a monad M, the authors in

Daniela Petrian and Ralph Sarkis. "Semialgebras and Weak Distributive Laws". In: *Proceedings MFPS 2021* ().

recently introduced the *semifree monad M^{s}*, with properties

- $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$ (M^{s}-algebras $\cong$ M-semialgebras)
- $\{\lambda : MT \overset{\text{w.d.l}}{\Rightarrow} TM\} \overset{\text{bijection}}{\leftrightarrow} \{\delta : M^{\mathrm{s}} T \overset{\text{d.l}}{\Rightarrow} TM^{\mathrm{s}} \text{ satisfying an extra condition}\}$.

Its definition is as follows:

$$M^{\mathrm{s}} := \mathsf{Id}_C + M,$$
$$\eta^{\mathrm{s}} := \mathsf{inl}^{\mathsf{Id}+M},$$
$$\mu^{\mathrm{s}} := [\mathsf{Id}_{\mathsf{Id}+M}, \mathsf{inr}^{\mathsf{Id}+M} \circ \mu \circ M[\eta, \mathsf{Id}_M]].$$

Semifree monads

Given a monad M, the authors in

Daniela Petrian and Ralph Sarkis. "Semialgebras and Weak Distributive Laws". In: *Proceedings MFPS 2021* ().

recently introduced the *semifree monad M^{s}*, with properties

- $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$ (M^{s}-algebras $\cong$ M-semialgebras)
- $\{\lambda : MT \overset{\text{w.d.l}}{\Rightarrow} TM\} \overset{\text{bijection}}{\leftrightarrow} \{\delta : M^{\mathrm{s}} T \overset{\text{d.l}}{\Rightarrow} TM^{\mathrm{s}} \text{ satisfying an extra condition}\}$.

Its definition is as follows:

$$M^{\mathrm{s}} := \mathsf{Id}_C + M,$$
$$\eta^{\mathrm{s}} := \mathsf{inl}^{\mathsf{Id}+M},$$
$$\mu^{\mathrm{s}} := [\mathsf{Id}_{\mathsf{Id}+M}, \mathsf{inr}^{\mathsf{Id}+M} \circ \mu \circ M[\eta, \mathsf{Id}_M]].$$

Question: Infer a presentation of M^{s} from a presentation of M?

Intuition

- $M^{\mathrm{s}}X = X + MX$

Intuition

- $M^{s}X = X + MX$
  - "new" set of variables = "left X"

Intuition

- $M^{\mathrm{s}}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{\mathrm{s}}X$

Intuition

- $M^{\mathrm{s}}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{\mathrm{s}}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).

Intuition

- $M^{s}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{s}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).
- Take algebra $M^{s}X \xrightarrow{\gamma} X$. Then $\gamma = [\mathsf{Id}_X, \alpha]$ with $MX \xrightarrow{\alpha} X$ semialgebra.

Intuition

- $M^{s}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{s}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).
- Take algebra $M^{s}X \xrightarrow{\gamma} X$. Then $\gamma = [\mathsf{Id}_X, \alpha]$ with $MX \xrightarrow{\alpha} X$ semialgebra.
- Notice:

$$(\alpha \cdot \eta_X) \cdot (\alpha \cdot \eta_X)$$

Intuition

- $M^{\mathrm{s}}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{\mathrm{s}}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).
- Take algebra $M^{\mathrm{s}}X \xrightarrow{\gamma} X$. Then $\gamma = [\mathsf{Id}_X, \alpha]$ with $MX \xrightarrow{\alpha} X$ semialgebra.
- Notice:

$$(\alpha \cdot \eta_X) \cdot (\alpha \cdot \eta_X) \overset{\eta \text{ nat.}}{=} \alpha \cdot M\alpha \cdot \eta_{MX} \cdot \eta_X$$

Intuition

- $M^{\mathrm{s}}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{\mathrm{s}}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).
- Take algebra $M^{\mathrm{s}}X \xrightarrow{\gamma} X$. Then $\gamma = [\mathrm{Id}_X, \alpha]$ with $MX \xrightarrow{\alpha} X$ semialgebra.
- Notice:

$$(\alpha \cdot \eta_X) \cdot (\alpha \cdot \eta_X) \overset{\eta \text{ nat.}}{=} \alpha \cdot M\alpha \cdot \eta_{MX} \cdot \eta_X \overset{\text{assoc. } \alpha}{=} \alpha \cdot \mu_X \cdot \eta_{MX} \cdot \eta_X$$

Intuition

- $M^{\mathrm{s}}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{\mathrm{s}}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).
- Take algebra $M^{\mathrm{s}}X \xrightarrow{\gamma} X$. Then $\gamma = [\mathrm{Id}_X, \alpha]$ with $MX \xrightarrow{\alpha} X$ semialgebra.
- Notice:

$$(\alpha \cdot \eta_X) \cdot (\alpha \cdot \eta_X) \overset{\eta \text{ nat.}}{=} \alpha \cdot M\alpha \cdot \eta_{MX} \cdot \eta_X \overset{\text{assoc. } \alpha}{=} \alpha \cdot \mu_X \cdot \eta_{MX} \cdot \eta_X \overset{\text{unit ax.}}{=} (\alpha \cdot \eta_X).$$

Intuition

- $M^{\mathrm{s}}X = X + MX$
  - "new" set of variables = "left X"
  - "old" set of variables = $\eta_X(X) \subseteq MX$, free of constraints in $M^{\mathrm{s}}X$
  - $\eta_X : X \to MX \quad \Rightarrow \quad$ new unary operation (a : 1).
- Take algebra $M^{\mathrm{s}}X \xrightarrow{\gamma} X$. Then $\gamma = [\mathrm{Id}_X, \alpha]$ with $MX \xrightarrow{\alpha} X$ semialgebra.
- Notice:

$$(\alpha \cdot \eta_X) \cdot (\alpha \cdot \eta_X) \overset{\eta \text{ nat.}}{=} \alpha \cdot M\alpha \cdot \eta_{MX} \cdot \eta_X \overset{\text{assoc. } \alpha}{=} \alpha \cdot \mu_X \cdot \eta_{MX} \cdot \eta_X \overset{\text{unit ax.}}{=} (\alpha \cdot \eta_X).$$

- This idempotent will be the interpretation of (a : 1).

Contribution

Definition
Given an algebraic theory (Σ, E), we define a new algebraic theory $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ by $\Sigma^{\mathrm{s}} := \Sigma \uplus \{\mathsf{a} : 1\}$ and E^{s} defined as:

$$\mathsf{a}\mathsf{a}v = \mathsf{a}v,$$

Contribution

Definition
Given an algebraic theory (Σ, E), we define a new algebraic theory $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ by $\Sigma^{\mathrm{s}} := \Sigma \uplus \{\mathsf{a} : 1\}$ and E^{s} defined as:

$$\begin{aligned} \mathsf{a}\mathsf{a}v &= \mathsf{a}v, \\ \mathsf{a}(\mathsf{op}(v_1, \dots, v_n)) &= \mathsf{op}(v_1, \dots, v_n), \\ \mathsf{op}(\mathsf{a}v_1, \dots, \mathsf{a}v_n) &= \mathsf{op}(v_1, \dots, v_n), \end{aligned}$$

Contribution

Definition
Given an algebraic theory (Σ, E), we define a new algebraic theory $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ by $\Sigma^{\mathrm{s}} := \Sigma \uplus \{\mathsf{a} : 1\}$ and E^{s} defined as:

$$\begin{aligned} \mathsf{a}\mathsf{a}v &= \mathsf{a}v, \\ \mathsf{a}(\mathsf{op}(v_1, \ldots, v_n)) &= \mathsf{op}(v_1, \ldots, v_n), \\ \mathsf{op}(\mathsf{a}v_1, \ldots, \mathsf{a}v_n) &= \mathsf{op}(v_1, \ldots, v_n), \\ t(\mathsf{a}v_1, \ldots, \mathsf{a}v_n) &= s(\mathsf{a}v_1, \ldots, \mathsf{a}v_n), \end{aligned}$$

for all $(\mathsf{op} : n) \in \Sigma$ and $(t(v_1, \ldots, v_n) = s(v_1, \ldots, v_n)) \in E$.

Contribution

Definition
Given an algebraic theory (Σ, E), we define a new algebraic theory $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ by $\Sigma^{\mathrm{s}} := \Sigma \uplus \{\mathsf{a} : 1\}$ and E^{s} defined as:

$$\begin{aligned} \mathsf{a}\mathsf{a}v &= \mathsf{a}v, \\ \mathsf{a}(\mathsf{op}(v_1, \dots, v_n)) &= \mathsf{op}(v_1, \dots, v_n), \\ \mathsf{op}(\mathsf{a}v_1, \dots, \mathsf{a}v_n) &= \mathsf{op}(v_1, \dots, v_n), \\ t(\mathsf{a}v_1, \dots, \mathsf{a}v_n) &= s(\mathsf{a}v_1, \dots, \mathsf{a}v_n), \end{aligned}$$

for all $(\mathsf{op} : n) \in \Sigma$ and $(t(v_1, \dots, v_n) = s(v_1, \dots, v_n)) \in E$.

Theorem
Let (M, η, μ) be a monad on Set*. If (Σ, E) is a concrete algebraic presentation of (M, η, μ) then $(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$ is a concrete algebraic presentation of the semifree monad $(M^{\mathrm{s}}, \eta^{\mathrm{s}}, \mu^{\mathrm{s}})$.*

Sketch of proof

Since $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$, it suffices to prove $\mathbf{EM}_s(M) \cong \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$.

Sketch of proof

Since $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M)$, it suffices to prove $\mathbf{EM}_s(M) \cong \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$.
We will define two functors and prove they are inverses:

$$G : \mathbf{EM}_s(M) \rightleftarrows \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) : H.$$

- Define G.
- Define H.
- Prove G and H are inverses.

Forward direction

Let

$$\begin{cases} G : \mathbf{EM}_s(M) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ G(X, \alpha) := (X, \langle\!|\cdot|\!\rangle) \\ G(f) := f \end{cases}$$

Forward direction

Let

$$\begin{cases} G : \mathbf{EM}_s(M) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ G(X, \alpha) := (X, \langle\!\langle \cdot \rangle\!\rangle) \\ G(f) := f \end{cases}$$

where the interpretation $\langle\!\langle \cdot \rangle\!\rangle$ is

$$\langle\!\langle \mathsf{a} \rangle\!\rangle := \left(X \xrightarrow{\eta_X} MX \xrightarrow{\alpha} X \right),$$

$$\langle\!\langle \mathsf{op} \rangle\!\rangle := \left(X^n \xrightarrow{(\eta_X)^n} (MX)^n \qquad\qquad MX \xrightarrow{\alpha} X \right),$$

Forward direction

Let

$$\begin{cases} G : \mathbf{EM}_s(M) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ G(X, \alpha) := (X, \langle\!\langle \cdot \rangle\!\rangle) \\ G(f) := f \end{cases}$$

where the interpretation $\langle\!\langle \cdot \rangle\!\rangle$ is

$$\langle\!\langle \mathsf{a} \rangle\!\rangle := \left(X \xrightarrow{\eta_X} MX \xrightarrow{\alpha} X \right),$$

$$\langle\!\langle \mathsf{op} \rangle\!\rangle := \left(X^n \xrightarrow{(\eta_X)^n} (MX)^n \xrightarrow{[\![\mathsf{op}]\!]^{MX}} MX \xrightarrow{\alpha} X \right),$$

where $(MX, [\![\cdot]\!]^{MX})$ is the image of (MX, μ_X) along the presentation.

Forward direction

Let

$$\begin{cases} G : \mathbf{EM}_s(M) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ G(X, \alpha) := (X, \langle\!|\cdot|\!\rangle) \\ G(f) := f \end{cases}$$

where the interpretation $\langle\!|\cdot|\!\rangle$ is

$$\langle\!|\mathsf{a}|\!\rangle := \left(X \xrightarrow{\eta_X} MX \xrightarrow{\alpha} X \right),$$

$$\langle\!|\mathsf{op}|\!\rangle := \left(X^n \xrightarrow{(\eta_X)^n} (MX)^n \xrightarrow{[\![\mathsf{op}]\!]^{MX}} MX \xrightarrow{\alpha} X \right),$$

where $(MX, [\![\cdot]\!]^{MX})$ is the image of (MX, μ_X) along the presentation.

Lemma: For all t of depth at least 1: $\langle\!| t |\!\rangle = \alpha \circ [\![t]\!]^{MX}_{\eta_X}$.

Forward direction

Let

$$\begin{cases} G : \mathbf{EM}_s(M) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ G(X, \alpha) := (X, \langle\!|\cdot|\!\rangle) \\ G(f) := f \end{cases}$$

where the interpretation $\langle\!|\cdot|\!\rangle$ is

$$\langle\!|\mathsf{a}|\!\rangle := \left(X \xrightarrow{\eta_X} MX \xrightarrow{\alpha} X\right),$$

$$\langle\!|\mathsf{op}|\!\rangle := \left(X^n \xrightarrow{(\eta_X)^n} (MX)^n \xrightarrow{[\![\mathsf{op}]\!]^{MX}} MX \xrightarrow{\alpha} X\right),$$

where $(MX, [\![\cdot]\!]^{MX})$ is the image of (MX, μ_X) along the presentation.

Lemma: For all t of depth at least 1: $\langle\!|t|\!\rangle = \alpha \circ [\![t]\!]^{MX}_{\eta_X}$.

Lemma: The interpretation $\langle\!|\cdot|\!\rangle$ satisfies the four equations in E^{s}.

Forward direction

Let

$$\begin{cases} G : \mathbf{EM}_s(M) \to \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \\ G(X, \alpha) := (X, \langle\!|\cdot|\!\rangle) \\ G(f) := f \end{cases}$$

where the interpretation $\langle\!|\cdot|\!\rangle$ is

$$\langle\!|\mathsf{a}|\!\rangle := \left(X \xrightarrow{\eta_X} MX \xrightarrow{\alpha} X \right),$$

$$\langle\!|\mathsf{op}|\!\rangle := \left(X^n \xrightarrow{(\eta_X)^n} (MX)^n \xrightarrow{[\![\mathsf{op}]\!]^{MX}} MX \xrightarrow{\alpha} X \right),$$

where $(MX, [\![\cdot]\!]^{MX})$ is the image of (MX, μ_X) along the presentation.

Lemma: For all t of depth at least 1: $\langle\!|t|\!\rangle = \alpha \circ [\![t]\!]^{MX}_{\eta_X}$.

Lemma: The interpretation $\langle\!|\cdot|\!\rangle$ satisfies the four equations in E^{s}.

Lemma: G maps homomorphisms to homomorphisms.

Backward direction

Let

$$\begin{cases} H : \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \to \mathbf{EM}_s(M) \\ H(X, \llparenthesis \cdot \rrparenthesis) := (X, \alpha) \\ H(f) := f \end{cases}$$

Backward direction

Let

$$\begin{cases} H : \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \to \mathbf{EM}_s(M) \\ H(X, \llparenthesis \cdot \rrparenthesis) := (X, \alpha) \\ H(f) := f \end{cases}$$

Defining $\alpha : MX \to X$ directly is difficult since MX is arbitrary. To solve this we use an isomorphism between free objects:

$$\left({}^{\mathcal{T}_\Sigma(X)}/_E, [\![\cdot]\!]\right) \underset{h^{-1}}{\overset{h}{\rightleftarrows}} (MX, [\![\cdot]\!]^{MX}).$$

Backward direction

Let

$$\begin{cases} H : \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \to \mathbf{EM}_s(M) \\ H(X, \langle\!|\cdot|\!\rangle) := (X, \alpha) \\ H(f) := f \end{cases}$$

Defining $\alpha : MX \to X$ directly is difficult since MX is arbitrary. To solve this we use an isomorphism between free objects:

$$\left({}^{\mathcal{T}_\Sigma(X)}/_{E}, [\![\cdot]\!]\right) \underset{h^{-1}}{\overset{h}{\rightleftarrows}} (MX, [\![\cdot]\!]^{MX}).$$

We define $\alpha : MX \to X : c \mapsto \langle\!| t |\!\rangle_{\langle\!|\mathsf{a}|\!\rangle}$, where $t \in h^{-1}(c)$.

Backward direction

Let

$$\begin{cases} H : \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \to \mathbf{EM}_s(M) \\ H(X, \llparenthesis\cdot\rrparenthesis) := (X, \alpha) \\ H(f) := f \end{cases}$$

Defining $\alpha : MX \to X$ directly is difficult since MX is arbitrary. To solve this we use an isomorphism between free objects:

$$\left(\mathcal{T}_\Sigma(X)/_E, [\![\cdot]\!]\right) \underset{h^{-1}}{\overset{h}{\rightleftarrows}} (MX, [\![\cdot]\!]^{MX}).$$

We define $\alpha : MX \to X : c \mapsto \llparenthesis t \rrparenthesis_{\llparenthesis \mathsf{a} \rrparenthesis}$, where $t \in h^{-1}(c)$.

Lemma: The function α is well-defined.

Backward direction

Let

$$\begin{cases} H : \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \to \mathbf{EM}_s(M) \\ H(X, \langle\!|\cdot|\!\rangle) := (X, \alpha) \\ H(f) := f \end{cases}$$

Defining $\alpha : MX \to X$ directly is difficult since MX is arbitrary. To solve this we use an isomorphism between free objects:

$$\left(\mathcal{T}_\Sigma(X)/_E, [\![\cdot]\!]\right) \underset{h^{-1}}{\overset{h}{\rightleftarrows}} (MX, [\![\cdot]\!]^{MX}).$$

We define $\alpha : MX \to X : c \mapsto \langle\!|t|\!\rangle_{\langle\!|\mathsf{a}|\!\rangle}$, where $t \in h^{-1}(c)$.

Lemma: The function α is well-defined.

Lemma: The function α satisfies associativity, i.e., is an M-semialgebra.

Backward direction

Let

$$\begin{cases} H : \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}}) \to \mathbf{EM}_s(M) \\ H(X, \langle\!|\cdot|\!\rangle) := (X, \alpha) \\ H(f) := f \end{cases}$$

Defining $\alpha : MX \to X$ directly is difficult since MX is arbitrary. To solve this we use an isomorphism between free objects:

$$\left(\mathcal{T}_\Sigma(X)/_E, [\![\cdot]\!]\right) \underset{h^{-1}}{\overset{h}{\rightleftarrows}} \left(MX, [\![\cdot]\!]^{MX}\right).$$

We define $\alpha : MX \to X : c \mapsto \langle\!| t |\!\rangle_{\langle\!|\mathsf{a}|\!\rangle}$, where $t \in h^{-1}(c)$.

Lemma: The function α is well-defined.

Lemma: The function α satisfies associativity, i.e., is an M-semialgebra.

Lemma: H maps homomorphisms to homomorphisms.

End of the proof

Summary: We want a concrete isomorphism $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$.
We have

$$\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M) \underset{H}{\overset{G}{\rightleftarrows}} \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$$

where G and H are well-defined concrete functors.

End of the proof

Summary: We want a concrete isomorphism $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$.
We have

$$\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M) \underset{H}{\overset{G}{\rightleftarrows}} \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$$

where G and H are well-defined concrete functors.

Lemma: The functors G and H are inverses.

End of the proof

Summary: We want a concrete isomorphism $\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$.
We have

$$\mathbf{EM}(M^{\mathrm{s}}) \cong \mathbf{EM}_s(M) \underset{H}{\overset{G}{\rightleftarrows}} \mathbf{Alg}(\Sigma^{\mathrm{s}}, E^{\mathrm{s}})$$

where G and H are well-defined concrete functors.

Lemma: The functors G and H are inverses.

The theorem is proved! □

Examples

- Maybe: $(- + 1)^{\mathrm{s}}$ has presentation
  $\Sigma^{\mathrm{s}} = \{\bullet : 0, \mathsf{a} : 1\}, E^{\mathrm{s}} = \{\mathsf{a}\mathsf{a}v = \mathsf{a}v,\ \mathsf{a}\bullet = \bullet\}$.

Examples

- Maybe: $(- + 1)^{\mathrm{s}}$ has presentation $\Sigma^{\mathrm{s}} = \{\bullet : 0, \mathsf{a} : 1\}, E^{\mathrm{s}} = \{\mathsf{a}\mathsf{a}v = \mathsf{a}v,\ \mathsf{a}\bullet = \bullet\}$.
- Distribution: $\mathcal{D}^{\mathrm{s}}$ has presentation $\Sigma^{\mathrm{s}}_{\mathcal{D}} = \{\mathsf{a} : 1\} \cup \{+_p : 2 \mid p \in (0,1)\}$,

$$E^{\mathrm{s}}_{\mathcal{D}} = \left\{ \begin{gathered} \mathsf{a}\mathsf{a}v = \mathsf{a}v, \\ \mathsf{a}(u +_p v) = u +_p v, \\ \mathsf{a}u +_p \mathsf{a}v = u +_p v, \\ v +_p v = \mathsf{a}v, \\ u +_p v = v +_{1-p} u, \\ (u +_q v) +_p w = u +_{pq} \left(v +_{\frac{p(1-q)}{1-pq}} w\right) \end{gathered} \right\}.$$

Examples

- Maybe: $(- + 1)^{\mathrm{s}}$ has presentation
  $\Sigma^{\mathrm{s}} = \{\bullet : 0, \mathsf{a} : 1\}, E^{\mathrm{s}} = \{\mathsf{aa}v = \mathsf{a}v,\ \mathsf{a}\bullet = \bullet\}$.
- Distribution: $\mathcal{D}^{\mathrm{s}}$ has presentation
  $\Sigma^{\mathrm{s}}_{\mathcal{D}} = \{\mathsf{a} : 1\} \cup \{+_p : 2 \mid p \in (0,1)\}$,

$$E^{\mathrm{s}}_{\mathcal{D}} = \left\{ \begin{gathered} \mathsf{aa}v = \mathsf{a}v, \\ \mathsf{a}(u +_p v) = u +_p v, \\ \mathsf{a}u +_p \mathsf{a}v = u +_p v, \\ v +_p v = \mathsf{a}v, \\ u +_p v = v +_{1-p} u, \\ (u +_q v) +_p w = u +_{pq} (v +_{\frac{p(1-q)}{1-pq}} w) \end{gathered} \right\}.$$

Indeed, the following rules generalise to terms of depth $\geqslant 1$:

$$\mathsf{a}(\mathsf{op}(v_1, \ldots, v_n)) = \mathsf{op}(v_1, \ldots, v_n) = \mathsf{op}(\mathsf{a}v_1, \ldots, \mathsf{a}v_n).$$

Conclusion

We have shown:

$$\text{concrete alg. pres. of } M \Rightarrow \text{concrete alg. pres. of } M^{\mathrm{s}}.$$

Conclusion

We have shown:

$$\text{concrete alg. pres. of } M \Rightarrow \text{concrete alg. pres. of } M^{\mathrm{s}}.$$

Related results:

- $M(- + 1)$ is a monad. Its presentation is the one of M with an additional constant and no additional equations.

Conclusion

We have shown:

$$\text{concrete alg. pres. of } M \Rightarrow \text{concrete alg. pres. of } M^{\mathrm{s}}.$$

Related results:

- $M(- + 1)$ is a monad. Its presentation is the one of M with an additional constant and no additional equations.
- Suppose TM is a monad because of a distr. law $\lambda : MT \Rightarrow TM$. Zwart and Marsden give a concrete presentation of TM using λ and concrete presentations of T and M in

Maaike Zwart and Dan Marsden. "No-Go Theorems for Distributive Laws". In: *34th Symposium on Logic in Computer Science (LICS)*. 2019.

Conclusion and future work

Further work:

- Infer new presentations for other monad constructions, e.g.

Conclusion and future work

Further work:

- Infer new presentations for other monad constructions, e.g.
    - coproduct of monads coproducts of monads [8],

Conclusion and future work

Further work:

- Infer new presentations for other monad constructions, e.g.
  - coproduct of monads coproducts of monads [8],
  - monad transformers [7],

Conclusion and future work

Further work:

- Infer new presentations for other monad constructions, e.g.
  - coproduct of monads coproducts of monads [8],
  - monad transformers [7],
  - iterated semifree construction.

Conclusion and future work

Further work:

- Infer new presentations for other monad constructions, e.g.
  - coproduct of monads coproducts of monads [8],
  - monad transformers [7],
  - iterated semifree construction.
- Infer presentations in other categories from presentations in Set.

Conclusion and future work

Further work:

- Infer new presentations for other monad constructions, e.g.
  - coproduct of monads coproducts of monads [8],
  - monad transformers [7],
  - iterated semifree construction.
- Infer presentations in other categories from presentations in Set.
- Obtain no-go theorems for weak distributive laws, analogous to Zwart and Marsden's work. (New no-go theorems are needed, because existing ones require reduction to variables)

References

[1] S. Awodey. *Category Theory*. Oxford Logic Guides. Ebsco Publishing, 2006.

[2] John Beck. "Distributive laws". In: *Seminar on triples and categorical homology theory* (1969), pp. 119–140.

[3] Richard Garner. "The Vietoris Monad and Weak Distributive Laws". In: *Applied Categorical Structures* 28.2 (Oct. 2019), pp. 339–354.

[4] Alexandre Goy and Daniela Petrian. "Combining Probabilistic and Non-Deterministic Choice via Weak Distributive Laws". In: LICS '20. 2020, pp. 454–464.

[5] Martin Hyland and Christine Tasson. "The linear-non-linear substitution 2-monad". In: *Electronic Proceedings in Theoretical Computer Science* 333 (Feb. 2021), pp. 215–229.

[6] Bart Jacobs. "Convexity, Duality and Effects". In: vol. 323. Aug. 2010, pp. 1–19. ISBN: 978-3-642-15239-9.

[7] Sheng Liang, Paul Hudak, and Mark Jones. "Monad Transformers and Modular Interpreters". In: *Proceedings of the 22nd ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages*. POPL '95. San Francisco, California, USA: Association for Computing Machinery, 1995, pp. 333–343.

[8] Christoph Lüth and Neil Ghani. "Composing monads using coproducts". In: vol. 37. Sept. 2002, pp. 133–144.

[9] Saunders MacLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics, Vol. 5. Springer-Verlag, 1971, pp. ix+262.

[10] Ernest Manes. *Algebraic Theories*. Vol. 26. Graduate Texts in Mathematics. Springer, Jan. 1976. ISBN: 978-1-4612-9862-5.

[11] Matteo Mio, Ralph Sarkis, and Valeria Vignudelli. "Combining nondeterminism, probability, and termination: equational and metric reasoning". In: *CoRR* abs/2012.00382 (2020).

Thanks

Thank you for your attention !